CIÊNCIA e NATURA

ISSN 2179-460X
OPEN ACCESS

Ci. e Nat., Santa Maria, v. 48, e94345, 2026• https://doi.org/10.5902/2179460X94345
Submitted: 06/11/2025 • Approved: 03/12/2025 • Published: 17/08/2026

Mathematics

Well-Posedness and homogenization applied to a one-dimensional elliptic problem

Boa postura e homogeneização aplicada a um problema elíptico unidimensional

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ABSTRACT

This paper studies the mathematical homogenization of a one-dimensional elliptic problem with a rapidly oscillating coefficient. The analysis focuses on the well-posedness of the problem, ensuring existence, uniqueness, and stability of the solution through the Lax–Milgram theorem and the coercivity and continuity of the associated bilinear form. Using the asymptotic homogenization method, an asymptotic expansion in two scales is derived, and the convergence between the exact and homogenized solutions is discussed, demonstrating $\mathcal{O}(\varepsilon)$ convergence order. An example with analytical solutions is provided to illustrate the theoretical results.

Keywords: One-dimensional problem with rapidly oscillating coefficient; Well-Posedness; Asymptotic homogenization method; Effective coefficient

RESUMO

Este artigo estuda a homogeneização matemática de um problema elíptico unidimensional com coeficiente rapidamente oscilante. A análise concentra-se na boa postura do problema, garantindo a existência, unicidade e estabilidade da solução por meio do teorema de Lax–Milgram e das condições de coercitividade e continuidade da forma bilinear associada. Utilizando o método de homogeneização assintótica, é obtida uma expansão assintótica em duas escalas, e discute-se a convergência entre as soluções exata e homogeneizada, demonstrando convergência de ordem $\mathcal{O}(\varepsilon)$. Um exemplo com soluções analíticas é fornecido para ilustrar os resultados teóricos.

Palavras-chave: Problema elíptico unidimensional com coeficiente rapidamente oscilante; Boa postura; Método de homogeneização assintótica; Coeficiente efetivo

1 INTRODUCTION

Heterogeneous media are those whose internal structure causes physical properties to vary with position. Examples of natural and artificial heterogeneous media include bones, soils, rocks, gels, ceramics, and composite materials. When such media exhibit separation of structural scales (Bakhvalov & Panasenko, 1989), which is described by a small geometric parameter $\varepsilon > 0$, they are called micro-heterogeneous. Furthermore, when these media have a periodic microstructure, meaning the medium is formed by the regular repetition of a basic cell, they are termed microperiodic.

Due to their diverse applications, the study of heterogeneous media is essential for scientific and technological advancement in various fields of knowledge, as their physical and geometric properties allow for the optimization of industrial processes and the development of more efficient materials in engineering, for example, to enhance techniques in medicine (Costa & dos Santos, 2010) and substance transport in porous media (Dimitrienko, 1997).

To study the influence of the microstructure on the effective macroscopic behavior of a micro-heterogeneous medium, mathematical homogenization is used. This technique produces an equivalent homogeneous medium whose constant properties are the effective properties of the heterogeneous medium. Mathematical homogenization brings together analytical techniques to obtain good approximations of the exact solutions of problems with rapidly oscillating coefficients, where the difference between the exact solution and its approximations is of the order of a power of ε in the norm of the function space where they are sought.

Among the classical methods, the asymptotic homogenization method (AHM) (Bakhvalov & Panasenko, 1989; Bensoussan et al., 1978; Ciouranescu & Donato, 1999; Tartar, 2009) stands out, used for problems modeling microperiodic media. Through the AHM, a formal asymptotic solution (FAS) is constructed as a two-scale power series in ε, whose first term represents the solution of the problem modeling the behavior of the equivalent homogeneous medium.

This work aims to study the well-posedness and apply the AHM to the following problem: given $\Omega = [0, 1] \subset \mathbb{R}$, for each ε, $0 < \varepsilon \ll 1$, find $u^\varepsilon \in C^2(\Omega)$ such that

$$\begin{cases} \mathcal{L}^{\varepsilon} u^{\varepsilon} \equiv \dfrac{d}{dx}\left[a^{\varepsilon}(x)\dfrac{du^{\varepsilon}}{dx}\right] = f(x) \ , \ x \in \Omega \\ \qquad\qquad u^{\varepsilon}(x) = 0 \ , \ x \in \partial\Omega, \end{cases} \tag{1}$$

where $a^{\varepsilon} \in C^1(\Omega)$ is ε-periodic, strictly positive and bounded, and $f \in C(\Omega)$.

Problem (1) can be interpreted, for instance, as modeling the temperature distribution u^{ε} in the steady–state thermal regime of a heat–conducting bar composed of a functionally graded material whose conductivity a^{ε} is ε-periodic. One-dimensional models of this type arise in practical applications such as composite materials, stratified media, and micro-engineered thermoelectric structures, where properties such as thermal conductivity vary rapidly along the domain. Although simplified, the one-dimensional configuration serves as a fundamental prototype for the development and rigorous analysis of homogenization methods, allowing for a detailed study of convergence rates and associated error estimates.

The main contributions of this work include a detailed well-posedness analysis establishing existence, uniqueness, and stability of solutions; the explicit construction of the effective coefficient through asymptotic homogenization; a convergence analysis proving $\mathcal{O}(\varepsilon)$ error estimates between exact and homogenized solutions; and an illustrative example validating the theoretical results.

2 METHODOLOGY

The methodology applies systematic procedures to the elliptic problem with periodic coefficients. The well-posedness analysis, in the sense of Hadamard, follows a structured approach involving the formulation of the variational problem in the Sobolev space $H_0^1(\Omega)$, the verification of the coercivity and continuity of the associated bilinear form, and the application of the Lax–Milgram theorem (Brezis, 2011) to establish existence and uniqueness. In addition, an a priori estimate based on maximum principles is employed to ensure the stability of the solution.

The asymptotic homogenization method follows a formal procedure that includes the construction of a two-scale FAS in powers of ε, the introduction of the fast variable $y = x/\varepsilon$ together with the corresponding derivative transformations via the chain rule, and the substitution of the FAS into the original problem. This leads to the systematic derivation of a hierarchy of cell problems by grouping terms according to their order in ε, the solution of these cell problems under the periodicity conditions ensured by Lemma 3, and the computation of the effective coefficient defining the homogenized equation, along with the guarantees of existence for the FAS.

The convergence analysis provides a rigorous justification of the method by estimating the residual obtained after substituting the FAS into the original equation, and by applying maximum-principle-based estimates to control the error. These arguments establish that the difference between the exact and homogenized solutions is of order $\mathcal{O}(\varepsilon)$ as $\varepsilon \to 0^+$. A numerical example is also presented to illustrate the theoretical results and to corroborate the predicted convergence rate.

3 PRELIMINARIES

Here, we present a collection of results needed for the subsequent analysis.

Theorem 1. *[Lax–Milgram theorem] Let H be a Hilbert space and H^* its dual. Let $A : H \times H \to \mathbb{R}$ be a bilinear form and $b : H^* \to \mathbb{R}$ a continuous linear functional. If the bilinear form A satisfies:*

1) *(Coercivity) there exists a constant $\alpha > 0$ such that*

$$A(u, u) \geq \alpha \|u\|_H^2, \quad \forall u \in H;$$

2) *(Continuity) there exists a constant $\beta > 0$ such that*

$$A(u, v) \leq \beta \|u\|_H \|v\|_H, \quad \forall u, v \in H;$$

then there exists a unique $u \in H$ such that

$$A(u, v) = b(v), \quad \forall v \in H.$$

Theorem 2. *[Estimate via maximum principle (Larsson & Thomée, 2009)] Let $\Omega \subset \mathbb{R}$ and $\mathcal{L}^\varepsilon$ be defined as in (1). Given $a^\varepsilon(x) \in C^1(\Omega)$ strictly positive and $u^\varepsilon \in C^2(\Omega)$, there exists a* *constant $K > 0$ such that*

$$\|u^\varepsilon\|_{C(\Omega)} \leq K\|f\|_{C(\Omega)}. \tag{2}$$

Lemma 3. *[Necessary and sufficient condition for the existence of periodic solutions (Bakhvalov & Panasenko, 1989)] Let $F(y)$ and $a(y)$ be 1-periodic functions in $y \in (0, n)$, $n \in \mathbb{N} \setminus \{1\}$, with $a(y)$ differentiable, strictly positive, and bounded. Then, a necessary and sufficient condition for the existence of a 1-periodic solution $N(y)$ to the equation $\mathcal{L}N = F$, where $\mathcal{L} \equiv \dfrac{d}{dy}\left[a(y)\dfrac{d}{dy}\right]$, is that $\langle F(y)\rangle = 0$, where $\langle\cdot\rangle \equiv \displaystyle\int_0^1 F(y)\,dy$ is the mean value operator. Such a 1-periodic solution is unique up to an additive constant, i.e., $N(y, C) = \tilde{N}(y) + C$ where $\tilde{N}(0) = 0$ or $\left\langle \tilde{N}(y)\right\rangle = 0$.*

Theorem 4. *[Cauchy–Schwarz inequality (Cioranescu et al., 2017)] Let $f, g \in L^2(\Omega)$, where $\Omega \subset \mathbb{R}$. Then*

$$|\,(f, g)\,| \leq \|f\|_{L^2(\Omega)}\,\|g\|_{L^2(\Omega)}, \tag{3}$$

where the inner product and the induced norm are given by

$$(f, g) = \int_\Omega f(x)g(x)\,dx, \qquad \|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2\,dx\right)^{1/2}.$$

Theorem 5. *[Poincaré inequality (Cioranescu et al., 2017)] Let $\Omega \subset \mathbb{R}$ be a bounded, and let $u \in H_0^1(\Omega)$. Then there exists a constant $C > 0$, depending only on Ω, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C\left\|\frac{du}{dx}\right\|^2_{L^2(\Omega)}. \tag{4}$$

4 RESULTS AND DISCUSSIONS

4.1 Well-posedness of the problem

Since $u^\varepsilon \in C_0^2(\Omega) \subset H_0^1(\Omega)$, the weak formulation of problem (1) is

$$\int_\Omega a^\varepsilon(x)\frac{du^\varepsilon}{dx}\frac{dw^\varepsilon}{dx}dx = \int_\Omega f(x)w^\varepsilon(x)dx\ ,\ \forall w^\varepsilon \in H_0^1(\Omega), \tag{5}$$

which can be written in terms of a bilinear form and a linear functional as

$$A(u^\varepsilon, w^\varepsilon) = b(w^\varepsilon), \tag{6}$$

where

$$A(u^\varepsilon, w^\varepsilon) = \left(a^\varepsilon(x) \frac{du^\varepsilon}{dx}, \frac{dw^\varepsilon}{dx} \right)_{L^2(\Omega)}, \qquad b(w^\varepsilon) = (f, w^\varepsilon)_{L^2(\Omega)}.$$

Let us prove that the bilinear form A satisfies the conditions of Theorem 1. Since the coefficient $a^\varepsilon(x)$ is bounded by hypothesis, there exists $K \in \mathbb{R}_+$ such that $|a^\varepsilon(x)| \leq K$. Then, by the Cauchy-Schwarz inequality in (3) and the norm equivalence between the H^1-induced norm on $H_0^1(\Omega)$ and the L^2-norm of the gradient

$$\begin{aligned} |A(u^\varepsilon, w^\varepsilon)| &= \left| \int_\Omega a^\varepsilon(x) \frac{du^\varepsilon}{dx} \frac{dw^\varepsilon}{dx} dx \right| \leq \left\| a^\varepsilon(x) \frac{du^\varepsilon}{dx} \right\|_{L^2(\Omega)} \left\| \frac{dw^\varepsilon}{dx} \right\|_{L^2(\Omega)} \\ &\leq K \left\| \frac{du^\varepsilon}{dx} \right\|_{L^2(\Omega)} \left\| \frac{dw^\varepsilon}{dx} \right\|_{L^2(\Omega)} \leq K \|u^\varepsilon\|_{H_0^1(\Omega)} \|w^\varepsilon\|_{H_0^1(\Omega)}, \quad \forall u^\varepsilon, w^\varepsilon \in H_0^1(\Omega). \end{aligned} \tag{7}$$

Hence, A is continuous with constant $\beta = K$.

For coercivity, let $a_{\text{min}} = \inf_{x \in \overline{\Omega}} \{a^\varepsilon(x)\}$ and note that by the Poincaré inequality (4), the norms $\|\cdot\|_{H_0^1(\Omega)}$ and $\left\|\frac{d}{dx}(\cdot)\right\|_{L^2(\Omega)}$ are equivalent in $H_0^1(\Omega)$. Then, it is hold that

$$\|u\|_{H_0^1(\Omega)} \leq \sqrt{C+1} \left\| \frac{du}{dx} \right\|_{L^2(\Omega)}, \ \forall u \in H_0^1(\Omega).$$

Then,

$$\begin{aligned} A(u^\varepsilon, u^\varepsilon) &= \int_\Omega a^\varepsilon(x) \left(\frac{du^\varepsilon}{dx} \right)^2 dx \geq a_{\text{min}} \int_\Omega \left(\frac{du^\varepsilon}{dx} \right)^2 dx = a_{\text{min}} \int_\Omega \left| \frac{du^\varepsilon}{dx} \right|^2 dx \\ &= a_{\text{min}} \left\| \frac{du^\varepsilon}{dx} \right\|_{L^2(\Omega)}^2 \geq \frac{a_{\text{min}}}{C+1} \|u^\varepsilon\|_{H_0^1(\Omega)}^2, \ \forall u^\varepsilon \in H_0^1(\Omega). \end{aligned} \tag{8}$$

Thus, the bilinear form is coercive with constant $\alpha = a_{\text{min}}/(C+1)$. From (7) and (8), Theorem 1 ensures the existence of a unique solution $u^\varepsilon(x)$ to (6), hence a unique weak solution to (5). Since by hypothesis the solution belongs to $C^2(\Omega)$ and the coefficients are sufficiently continuously differentiable, $u^\varepsilon(x)$ is also a strong solution of (1).

For stability, let $\phi^\varepsilon(x)$ and $\varphi^\varepsilon(x)$ be solutions corresponding to source terms $f(x)$ and $g(x)$ in $C(\Omega)$, respectively. Then, for all $w^\varepsilon \in H_0^1(\Omega)$,

$$A(\phi^\varepsilon - \varphi^\varepsilon, w^\varepsilon) = \int_\Omega (f-g) w^\varepsilon dx.$$

Choosing $w^\varepsilon(x) = \phi^\varepsilon(x) - \varphi^\varepsilon(x)$, and applying coercivity, the Poincaré and Cauchy–Schwarz inequalities, and the norm equivalence in $H_0^1(\Omega)$, we get

$$\begin{aligned}\alpha \left\| \frac{d}{dx}(\phi^\varepsilon - \varphi^\varepsilon) \right\|^2_{L^2(\Omega)} &\leq \alpha \|\phi^\varepsilon - \varphi^\varepsilon\|^2_{H_0^1(\Omega)} \leq A(\phi^\varepsilon - \varphi^\varepsilon, \phi^\varepsilon - \varphi^\varepsilon) = \int_\Omega (f-g)(\phi^\varepsilon - \varphi^\varepsilon) dx \\ &\leq \left| \int_\Omega (f-g)(\phi^\varepsilon - \varphi^\varepsilon) dx \right| \leq \|f-g\|_{L^2(\Omega)} \|\phi^\varepsilon - \varphi^\varepsilon\|_{L^2(\Omega)} \\ &\leq C \|f-g\|_{L^2(\Omega)} \left\| \frac{d}{dx}(\phi^\varepsilon - \varphi^\varepsilon) \right\|_{L^2(\Omega)}.\end{aligned}$$

Simplifying, we obtain

$$\left\| \frac{d}{dx}(\phi^\varepsilon - \varphi^\varepsilon) \right\|_{L^2(\Omega)} \leq \frac{C}{\alpha} \|f-g\|_{L^2(\Omega)},$$

and by Poincaré's inequality again,

$$\|\phi^\varepsilon - \varphi^\varepsilon\|_{L^2(\Omega)} \leq \frac{C^2}{\alpha} \|f-g\|_{L^2(\Omega)}.$$

This shows that small changes in the source term lead to small changes in the solution, establishing stability and hence well-posedness in the sense of Hadamard.

4.2 Application of the AHM

The AHM proposes a formal asymptotic solution (FAS) of (1) as (Bakhvalov & Panasenko, 1989)

$$u^{(\infty)}(x, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i N_i\left(\frac{x}{\varepsilon}\right) \frac{d^i u}{dx^i} = \sum_{i=0}^{\infty} \varepsilon^i N_i(y) \frac{d^i u}{dx^i}, \tag{9}$$

where $y = x/\varepsilon$ and $N_i \in C^2(0, n)$ are 1-periodic functions, $i \in \{0\} \cup \mathbb{N}$. The regularity assumption $N_i \in C^2$ is imposed by the conditions satisfied by the original solution u^ε and is justified by the regularity of $a(y)$. In addition to these considerations, the lemma 3 also ensures that this regularity is satisfied. Also, it is assumed that there exist $u_j \in C^2(\Omega)$, $j \in \{0\} \cup \mathbb{N}$ such that the following asymptotic expansion holds:

$$u(x,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j u_j(x)\,. \tag{10}$$

With the new variable y, it follows that $a^\varepsilon(x) = a(x/\varepsilon) = a(y)$ must be 1-periodic, which implies that the functions $N_i(y)$ must also be 1-periodic.

To ensure the existence of (9) with the functions $N_i(y)$, these must satisfy $N_0(y) \equiv 1$ and

$$\mathcal{L}N_i - T_i(y) = h_i,\ i \geq 1, \tag{11}$$

where, with constants h_i, $i \geq 1$,

$$T_1(y) = -\frac{da}{dy}\,, \qquad T_i(y) = -a(y)\left(\frac{dN_{i-1}}{dy} + N_{i-2}(y)\right) - \frac{d}{dy}\left(a(y)N_{i-1}(y)\right)\,, i \geq 2\,, \tag{12}$$

which are obtained by substituting the FAS from (9) into (1), applying the differential operators $\mathcal{L}^\varepsilon$ and $\mathcal{L}$, taking into account the chain rule, and performing algebraic manipulations to obtain an expression with only one summation whose coefficients do not depend on ε.

To guarantee the existence of the 1-periodic solutions $N_i(y)$ of (11), Lemma 3 is employed, from which it follows that

$$h_i = -\left\langle T_i(y)\right\rangle\,, i \geq 1\,. \tag{13}$$

Furthermore, $N_i(0) = 0$, $i \geq 1$, is imposed as a uniqueness condition for these solutions.

Under these considerations, it follows that

$$\mathcal{L}^\varepsilon u^\varepsilon \sim \sum_{i=2}^{\infty} \varepsilon^{i-2} h_i \frac{d^i u}{dx^i}\,. \tag{14}$$

Then, (14) is an asymptotic expansion of the source term $f(x)$. For this, the function $u(x)$ given in (10) must satisfy the conditions

$$\begin{cases} \hat{a}\dfrac{d^2u_q}{dx^2} = f_q(x)\,,\ x \in \Omega \\ u_q(x) = 0\,,\ x \in \partial\Omega \end{cases} \tag{15}$$

where $\hat{a} = h_2 = \langle 1/a(y)\rangle^{-1}$ given explicitly as

$$\hat{a} = a(y) + a(y)\frac{dN_1}{dy}, \tag{16}$$

is the effective coefficient calculated via (13) with (12) for $i = 2$, and

$$f_q(x) = \begin{cases} f(x)\,,\ q = 0 \\ -\displaystyle\sum_{j=0}^{q-1} h_{q-j+2}\frac{d^{q-j+2}u_j}{dx^{q-j+2}}\,,\ q > 0. \end{cases} \tag{17}$$

Thus, from (9) - (17) with $q = 0$, it follows that the FAS has the form

$$u^{(\infty)}(x,\varepsilon) = u_0(x) + \sum_{i=1}^{\infty}\varepsilon^i N_i\left(\frac{x}{\varepsilon}\right)\frac{d^iu_0}{dx^i}, \tag{18}$$

where $u_0(x)$ is the solution of the homogenized problem, given by

$$\begin{cases} \hat{a}\dfrac{d^2u_0}{dx^2} = f(x)\,,\ x \in \Omega \\ u_0(x) = 0\,,\ x \in \partial\Omega \end{cases}. \tag{19}$$

The effective coefficient $\hat{a} = \langle 1/a(y)\rangle^{-1}$ represents the harmonic mean of the oscillatory coefficient $a(y)$, which characterizes the macroscopic behavior of the heterogeneous medium.

4.3 Convergence study

To study the convergence, a truncation of the FAS constructed in (18) up to order $\mathcal{O}(\varepsilon^2)$ is chosen. This choice is based on the fact that this truncation allows the construction of the homogenized problem in (19) and, not only that, it is the FAS with the smallest number of terms with this feature.

Thus, let the truncation of (18) up to order $\mathcal{O}(\varepsilon^2)$ be

$$u^{(2)}(x,\varepsilon) = u_0(x) + \varepsilon N_1(y)\frac{du_0}{dx} + \varepsilon^2 N_2(y)\frac{d^2u_0}{dx^2}, \tag{20}$$

then with

$$u^{(1)}(x,\varepsilon) = u_0(x) + \varepsilon N_1(y)\frac{du_0}{dx}, \tag{21}$$

it follows from assuming $u^{(1)}(x,\varepsilon)$ as a solution of (1) and using the equations in (11), (12) and (13) for $i=2$, $q=0$ and the equation of the homogenized problem (19), that

$$\mathcal{L}^\varepsilon u^{(1)} - f = -\varepsilon a(y)N_1(y)\frac{d^3u_0}{dx^3} - \frac{d}{dy}\left(a(y)N_1(y)\right)\frac{d^2u_0}{dx^2}. \tag{22}$$

Assuming $u^{(2)}(x,\varepsilon)$ as a solution of (1), it follows from (22)

$$\begin{aligned}
\mathcal{L}^\varepsilon u^{(2)} - f &= -\frac{d}{dx}\left[a(y)\frac{du^{(2)}}{dx}\right] - f = -\frac{d}{dx}\left[a(y)\frac{d}{dx}\left\{u^{(1)} + \varepsilon^2 N_2(y)\frac{d^2u_0}{dx^2}\right\}\right] - f \\
&= \mathcal{L}^\varepsilon u^{(1)} - f - \varepsilon^2\frac{d}{dx}\left[a(y)\frac{d}{dx}\left(N_2(y)\frac{d^2u_0}{dx^2}\right)\right] \\
&= -\varepsilon\left(h + a(y)\frac{dN_2}{dy}\right) - \varepsilon^2 a(y)N_2(y)\frac{d^4u_0}{dx^4},
\end{aligned}$$

where $h = a(y)dN_2/dy + a(y)N_1(y)$, which is constant by condition (13).

Defining

$$F^\varepsilon(x) = -\varepsilon\left(h + a\left(y\right)\frac{dN_2}{dy}\right)\frac{d^3u_0}{dx^3} - \varepsilon^2 a(y)N_2(y)\frac{d^4u_0}{dx^4},$$

we have that the problem for $u^{(2)}(x,\varepsilon)$ in terms of the differential operator of the problem in (1)

$$\begin{cases}
\mathcal{L}^\varepsilon u^{(2)} - f = F^\varepsilon(x), \ x \in \Omega \\
u^{(2)}(x,\varepsilon) = 0, \ x \in \partial\Omega
\end{cases}$$

for which, regarding the boundary conditions, we use the behavior of the solution $u_0(x)$ on the boundary and the uniqueness condition for the solutions of the local problems, $N_i(0) = 0$, $i = 1, 2$.

Thus, from the original problem, we define the problem

$$\begin{cases} \mathcal{L}^{\varepsilon}\left(u^{\varepsilon}-u^{(2)}\right)=-F^{\varepsilon},\ x\in\Omega \\ \left(u^{\varepsilon}-u^{(2)}\right)(x)=0,\ x\in\partial\Omega. \end{cases} \tag{23}$$

Therefore, for the problem in (23), the estimate in (2) holds, i.e.,

$$\left\|u^{\varepsilon}-u^{(2)}\right\|_{C(\Omega)}\le K\|F^{\varepsilon}\|_{C(\Omega)}. \tag{24}$$

By construction, the functions involved in the definition of F^{ε} are sufficiently continuously differentiable in Ω, therefore by the Weierstrass Theorem (Rudin, 1991), there exist constants $a_k\in\mathbb{R}_+^*$, $k\in\{1,2,3,4,5\}$ such that

$$|a(y)|\le a_1,\quad |N_2|\le a_2,\quad \left|\frac{dN_2}{dy}\right|\le a_3,\quad \left|\frac{d^3u_0}{dx^3}\right|\le a_4\quad\text{and}\quad \left|\frac{d^4u_0}{dx^4}\right|\le a_5. \tag{25}$$

Then, using $y=x/\varepsilon$ it follows from (25) that

$$\begin{aligned}\|F^{\varepsilon}\|_{C(\Omega)}&=\left\|-\varepsilon\left(h+a\left(y\right)\frac{dN_2}{dy}\right)\frac{d^3u_0}{dx^3}-\varepsilon^2a(y)N_2(y)\frac{d^4u_0}{dx^4}\right\|\\&\le\varepsilon\left\|\left(h+a(y)\frac{dN_2}{dy}\right)\frac{d^3u_0}{dx^3}\right\|+\varepsilon^2\left\|a(y)N_2(y)\frac{d^4u_0}{dx^4}\right\|\\&\le\varepsilon a_4(|h|+a_1a_3)+\varepsilon^2a_1a_2a_5\le\varepsilon\left[a_4(|h|+a_1a_3)+a_1a_2a_5\right]=\varepsilon K_1.\end{aligned}$$

Therefore, from (24)

$$\|u^{\varepsilon}-u^{(2)}\|_{C(\Omega)}\le KK_1\varepsilon=K_2\varepsilon. \tag{26}$$

Since $u^{(2)}-u_0=\varepsilon N_1\dfrac{du_0}{dx}+\varepsilon^2N_2\dfrac{d^2u_0}{dx^2}$, it follows analogously that there exists a constant $K_3>0$ such that

$$\|u^{(2)}-u_0\|_{C(\Omega)}\le K_3\varepsilon. \tag{27}$$

Thus, from (26) and (27)

$$\|u^{\varepsilon}-u_0\|_{C(\Omega)}\le\|u^{\varepsilon}-u^{(2)}\|_{C(\Omega)}+\|u^{(2)}-u_0\|_{C(\Omega)}=K_2\varepsilon+K_3\varepsilon=\varepsilon(K_2+K_3).$$

Therefore, $u^\varepsilon - u_0 = \mathcal{O}(\varepsilon)$ when $\varepsilon \to 0^+$. The constant $K_2 + K_3$ depends on the regularity of u_0 and the bounds of the local functions N_i and their derivatives, but is independent of ε.

It is worth noting that, in the one-dimensional case, a classical energetic approach yields the convergence $u^\varepsilon \to u_0$ in $H^1(\Omega)$ with the rate $O(\varepsilon^{1/2})$. This rate reflects the variational nature of the H^1 estimate, whose norm structure differs from the uniform norm used here. In contrast, under the additional continuity assumptions adopted in our analysis and using maximum principle based bounds together with uniform estimates for the local functions, the asymptotic homogenization method leads to an $O(\varepsilon)$ convergence rate in the $C(\Omega)$ norm. Although energetic arguments provide a shorter proof, we chose to keep the derivation fully within the classical asymptotic homogenization framework, while still obtaining results compatible with and in some aspects sharper than the standard energetic estimates; see, for example, (Ciouranescu & Donato, 1999).

4.4 An illustrative example

Consider the original problem (1) with $f(x) = -1$ and coefficient

$$a^\varepsilon(x) = 1 + \frac{1}{4}\cos\frac{2\pi x}{\varepsilon}, \tag{28}$$

whose rapidly oscillating behavior for decreasing values of $\varepsilon = 2^{-k}$, $k \in \{0,1,2,3\}$, is illustrated in Figure 1 together with the effective coefficient $\widehat{a} = \langle 1/a(y)\rangle^{-1} = \sqrt{15}/4$:

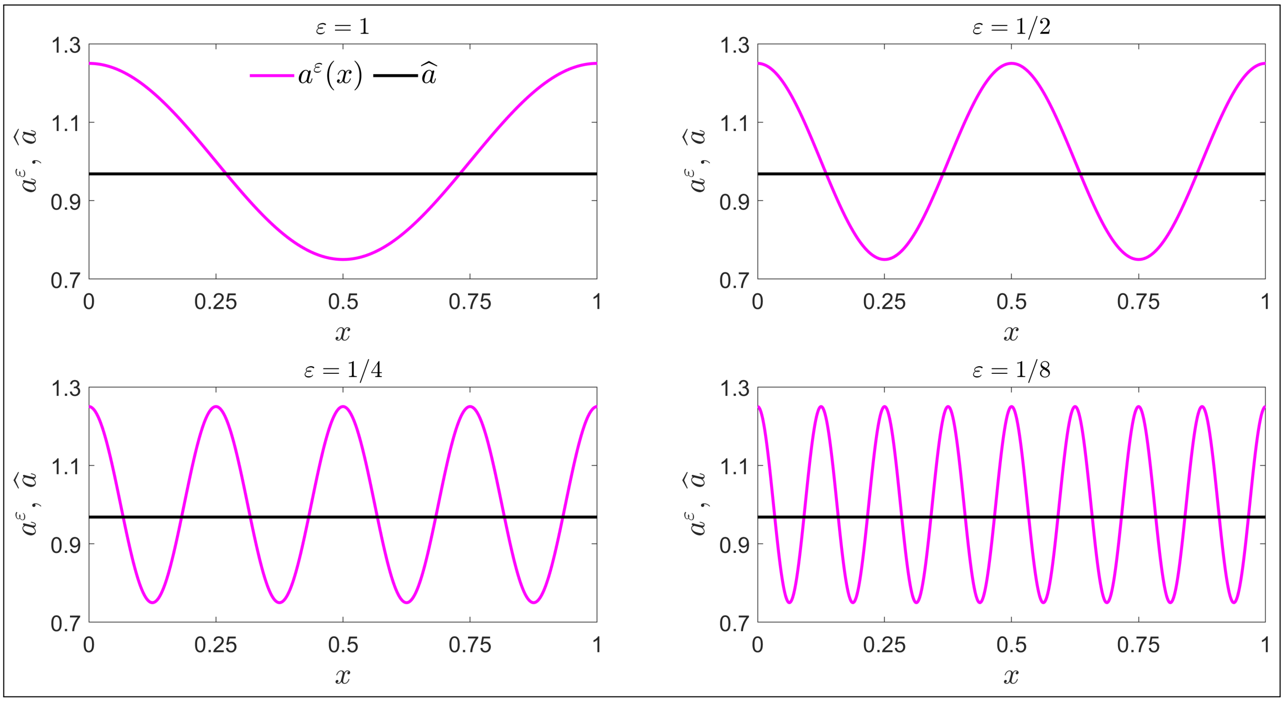
Problem (1) admits exact analytical solution via direct integration as

$$\begin{aligned} u^\varepsilon(x) &= \int_0^x \frac{1}{a^\varepsilon(t)}\left[\int_0^t f(s)ds - \left(\int_0^1 \frac{dr}{a^\varepsilon(r)}\right)^{-1}\int_0^1 \frac{1}{a^\varepsilon(r)}\int_0^r f(s)dsdr\right]dt \\ &= \left(1 - x - \frac{1}{A^\varepsilon(1)}\int_0^1 A^\varepsilon(t)dt\right)A^\varepsilon(x) + \int_0^x A^\varepsilon(t)dt \end{aligned}, \tag{29}$$

where $A^\varepsilon(x)$ is given by

$$A^\varepsilon(x) = \int_0^x \frac{dt}{a^\varepsilon(t)} = \begin{cases} v^\varepsilon(x)\,, & x \in \left(0, \dfrac{\varepsilon}{2}\right) \\ \dfrac{\varepsilon}{2\widehat{a}}\,, & x = \dfrac{\varepsilon}{2} \\ \dfrac{\varepsilon}{\widehat{a}} + v^\varepsilon(x)\,, & x \in \left(\dfrac{\varepsilon}{2}, 1\right) \end{cases}, \quad v^\varepsilon(x) = \frac{\varepsilon}{\widehat{a}\pi}\arctan\left(\frac{3}{4\widehat{a}}\tan\frac{\pi x}{\varepsilon}\right). \tag{30}$$

Figure 1 – Coefficient $a^\varepsilon(x)$ in (28) and effective coefficient $\widehat{a} = \sqrt{15}/4$



Source: the authors (2025)

The solution of the homogenized problem (19) is also obtained via direct integration as

$$u_0(x) = \frac{1}{2\widehat{a}} x(x-1) \; . \tag{31}$$

The first local problem for $N_1(y)$ is stated by (11) for $i = 1$ with $(12)_1$ and (13) for $i = 1$ together with condition $N_1(0) = 0$, and its solution is obtained analytically via direct integration as the periodic extension to $(0, n)$ of

$$N_1(y) = \begin{cases} n_1(y) \; , & y \in \left[0, \dfrac{1}{2}\right) \\ 0 \; , & y = \dfrac{1}{2} \\ 1 + n_1(y) \; , & y \in \left(\dfrac{1}{2}, 1\right) \end{cases} \; , \quad n_1(y) = \frac{1}{\pi} \arctan\left(\frac{3}{4\widehat{a}} \tan \pi y\right) - y \; . \tag{32}$$

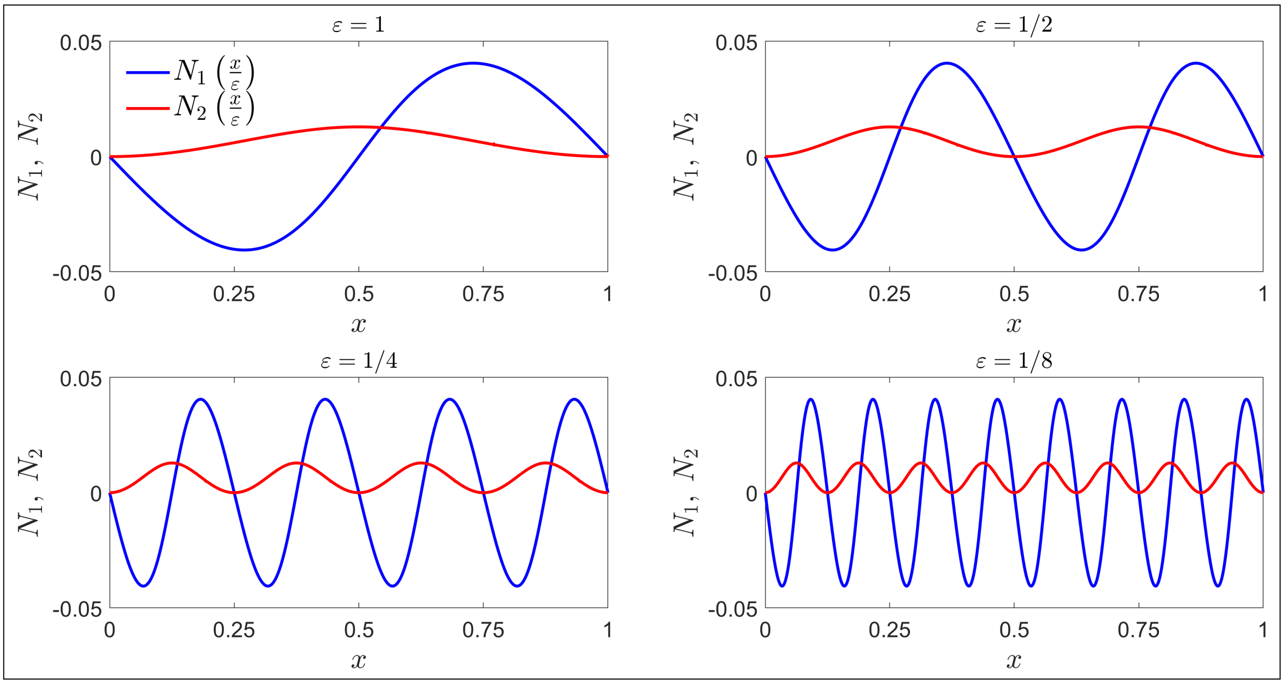
Note that $N_1(y)$ is antisymmetric with respect to $y = 1/2$, so $\langle N_1(y) \rangle = 0$.

In addition, the second local problem for $N_2(y)$ is stated by (11) with $(12)_2$ and (13) for $i = 2$ together with condition $N_2(0) = 0$, and its solution is obtained analytically via direct integration as the periodic extension to $(0, n)$ of

$$N_2(y) = -\int_0^y N_1(s)ds = \begin{cases} \displaystyle\int_0^y n_1(s)ds \ , & y \in \left[0, \frac{1}{2}\right] \\ \displaystyle\frac{1}{2} - y + \int_0^y n_1(s)ds \ , & y \in \left(\frac{1}{2}, 1\right) \end{cases} . \tag{33}$$

The rapidly oscillating behavior of the local functions $N_1(y)$ in (32) and $N_2(y)$ in (33) with respect to the macroscale for decreasing values of $\varepsilon = 2^{-k}$, $k \in \{0, 1, 2, 3\}$, is illustrated in Figure 2 below:

Figure 2 – Local functions $N_1(x/\varepsilon)$ in (32) and $N_2(x/\varepsilon)$ in (33)

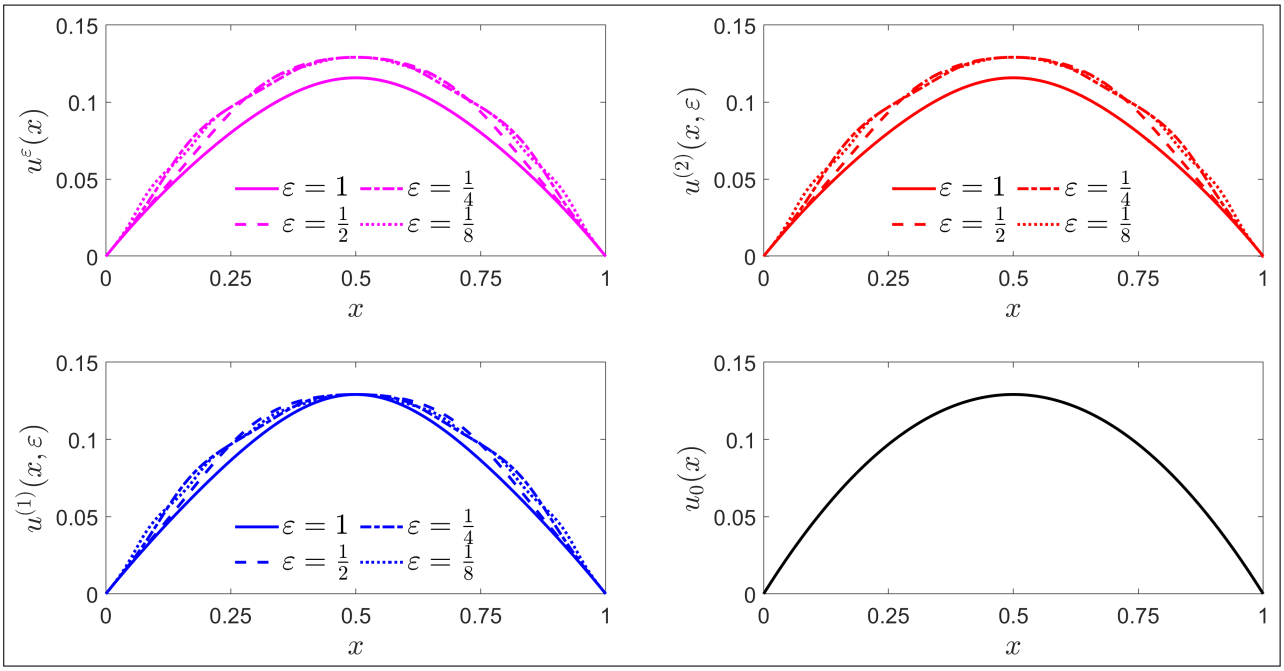


Source: the authors (2025)

With such considerations, FASs $u^{(1)}(x, \varepsilon)$ in (21) and $u^{(2)}(x, \varepsilon)$ in (20) can be completely determined, and their behaviors for decreasing values of $\varepsilon = 2^{-k}$, $k \in \{0, 1, 2, 3\}$, are shown individually in Figure 3 together with the exact solution $u^\varepsilon(x)$ in (29) and the homogenized solution $u_0(x)$ in (31).

In addition, Figure 4 shows how the exact solution $u^\varepsilon(x)$ and FASs $u^{(1)}(x, \varepsilon)$ and $u^{(2)}(x, \varepsilon)$ converge to the homogenized solution $u_0(x)$ as the value of ε decreases. As some of the curves become practically indistinguishable, Table 1 below presents the

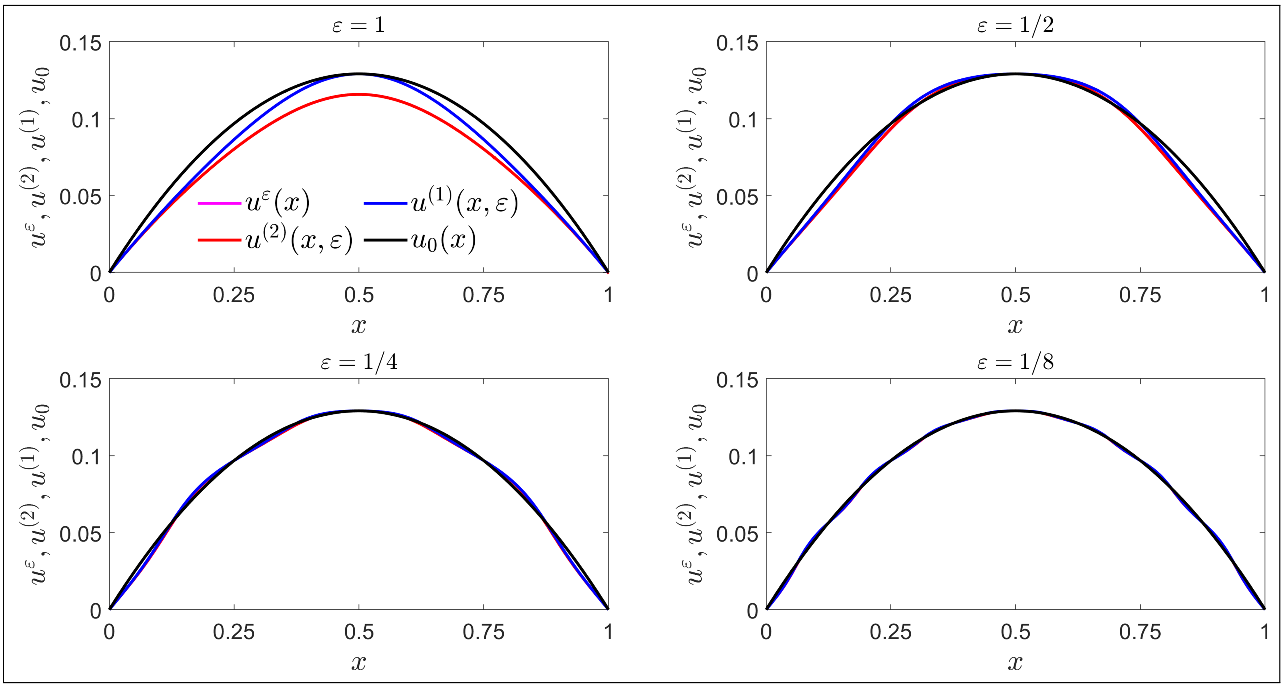
Figure 3 – Individual behaviors of $u^{\varepsilon}(x)$, $u^{(2)}(x,\varepsilon)$, $u^{(1)}(x,\varepsilon)$and $u_0(x)$ as ε decreases



Source: the authors (2025)

maximum absolute errors of the three approximations of u^{ε} ($u^{(2)}(x)$, $u^{(1)}(x)$ and $u_0(x)$) from best to worst.

Figure 4 – Convergence of u^{ε} ($u^{(2)}(x)$, $u^{(1)}(x)$ to $u_0(x)$ as ε decreases



Source: the authors (2025)

Table 1 – Maximum absolute errors of the three approximations of $u^\varepsilon(x)$

ε	$\max\limits_{x\in[0,1]} \lvert u^\varepsilon(x) - u^{(2)}(x,\varepsilon)\rvert$	$\max\limits_{x\in[0,1]} \lvert u^\varepsilon(x) - u^{(1)}(x,\varepsilon)\rvert$	$\max\limits_{x\in[0,1]} \lvert u^\varepsilon(x) - u_0(x)\rvert$
1	0.000516220400482	0.013315442077552	0.016721622978665
$1/2$	0.000143031593521	0.003462071603509	0.009477407873025
$1/4$	0.000200926828208	0.001031878188404	0.005061152636214
$1/8$	0.000234090029082	0.000440675358692	0.002654991758873

Source: the authors (2025)

The numerical results confirm the theoretical convergence rate of $\mathcal{O}(\varepsilon)$, with the second-order approximation $u^{(2)}$ providing the most accurate results across all values of ε.

5 CONCLUSION

This work has presented a detailed and self-contained study of the well-posedness and homogenization of a one-dimensional elliptic problem with rapidly oscillating coefficients. Beyond offering a general perspective on the role of homogenization in multiscale modeling, the results obtained here address, in a precise and explicit manner, the specific contributions highlighted throughout the article.

First, we established the well-posedness of the original heterogeneous problem. Using the Lax–Milgram theorem, we proved the existence and uniqueness of weak solutions, and by applying maximum–principle–based estimates we obtained stability and regularity properties compatible with the asymptotic framework. The coercivity and continuity of the associated bilinear form were explicitly derived, ensuring that the mathematical formulation is robust and applicable to a broad class of heterogeneous media.

Second, we derived the effective (homogenized) coefficient, which corresponds to the harmonic mean of the oscillatory coefficient. This coefficient arises from the solution of the periodic cell problem and encapsulates the macroscopic behavior induced by microscopic oscillations. Its explicit form highlights the physical relevance of the model, particularly in applications where the transport or diffusion properties depend critically on fine-scale fluctuations.

Third, we carried out a rigorous convergence analysis, proving that the asymptotic approximation converges to the exact solution with order $\mathcal{O}(\varepsilon)$ in the uniform norm. An illustrative example with analytical solutions confirmed this rate and showed that the second-order corrector significantly improves the approximation. This result not only validates the theoretical framework but also demonstrates its practical value for problems in which directly resolving the microscale would be computationally prohibitive.

Finally, the methodological structure developed here, connecting well-posedness, explicit homogenized coefficients, and quantitative convergence estimates provides a coherent foundation for extending the analysis to more complex and higher-dimensional models. Such extensions are relevant in applications involving heat conduction, elasticity, porous media flow, and other physical processes governed by heterogeneous materials. By providing a transparent and rigorous treatment of the one-dimensional case, this work contributes both to the theoretical understanding of homogenization and to its practical applicability in multiscale modeling.

ACKNOWLEDGEMENTS

Financial support provided by projects FAPITEC/SE Universal Nº 019203.01702/2024-6 and UNAM/PAPIIT Nº IN102026 is gratefully acknowledged. The authors also express their gratitude to the reviewers for the insightful comments, corrections, and constructive suggestions, which substantially improved the rigor, clarity, and overall presentation of this manuscript.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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How to cite this article

Machado, D. S., & Pérez-Fernández, L. D., & Bravo-Castillero J. (2026). Well-Posedness and homogenization applied to a one-dimensional elliptic problem. *Ciência e Natura*, Santa Maria, v. 48, e94345. DOI: https://doi.org/10.5902/2179460X94345.