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Mathematics

Comparative study of numerical inversion methods of the Laplace transform based on polynomial expansions and applied to the solution of a parabolic partial differential equation

Estudo comparativo de métodos de inversão numérica da transformada de Laplace baseados em expansões polinomiais e aplicados à solução de uma equação diferencial parcial parabólica

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ABSTRACT

The main objective of this article is to present and discuss the computational aspects of three algorithms based on polynomial expansions: Weeks, Gaussian Quadrature and Power Series, applied to Laplace Transform numerical inversion, to obtain a solution for a Parabolic Partial Differential Equation. In the methods evaluation process, the generated numerical results were compared to those obtained analytically, considering different values for the variables x and t. The Mean Absolute Error metric was used to estimate the optimal number of terms (N_o) in the sum, which also serves as an indicative parameter of the level of precision that can be obtained in each of the techniques considered. In general, all methods demonstrated good computational efficiency and accuracy in Laplace Transform inversion.

Keywords: Parabolic differential equation; Laplace transform; Numerical inverse; Polynomial expansions

RESUMO

Este artigo tem como objetivo apresentar e discutir os aspectos computacionais de três algoritmos baseados em expansões polinomiais: Weeks, Quadratura Gaussiana e Série de Potências, aplicados à inversão numérica da Transformada de Laplace para a obtenção da solução de uma Equação Diferencial Parcial Parabólica. No processo de avaliação dos métodos, os resultados numéricos gerados foram comparados aos obtidos analiticamente, considerando diferentes valores para as variáveis x e t. Empregou-se a métrica do Erro Absoluto Médio para a estimativa do número ótimo de termos (N_o) no

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somatório, a qual também se configura como um parâmetro indicativo do nível de precisão que pode ser obtido em cada uma das técnicas consideradas. De modo geral, todos os métodos demonstraram boa eficiência computacional e precisão na inversão da Transformada de Laplace.

Palavras-chave: Equação diferencial parabólica; Transformada de Laplace; Inversa numérica; Expansões polinomiais

1 INTRODUCTION

Differential Equations (DEs) are the mathematical foundation for various areas of Science and Engineering (Zill & Cullen, 2001) and arise from attempts to formulate or describe physical phenomena mathematically. These equations involve derivatives of an unknown function and can be classified as ordinary (ODEs), when there is only one independent variable, or partial (PDEs), when there is more than one (Zill & Cullen, 2001). Among PDEs there are Parabolic Partial Differential Equations (PPDEs), present in applications such as the study of temperature variations in soil (Lima & Araújo, 2019), in problems involving particle diffusion (Crank, 1975) and in fluid mechanics (Tani, 1977).

The Laplace Transform (LT) stands out as one of the methods used to solve PDEs. As presented by Pacheco (2011), the LT can be characterized by applying an integral operator to a function f, generally dependent on time t. Through this procedure, f is brought to a new domain, called the "Laplace domain", where the dependence becomes in terms of the variable s and the algebraic manipulations are typically simpler.

In particular, when using LT to solve physical problems mathematically modeled by PPDEs, the corresponding solutions are also written as a function of the variable s (Amaral, 2016). In this case, it is necessary to return to the original domain by inverting the Laplace Transform. However, depending on the complexity of the problem being studied, analytical LT inversion becomes unfeasible. From this need and the search for increasingly accurate results, numerical LT inversion methods emerge (Amaral, 2016; Da Silva et al., 2025). There are different numerical inversion techniques in the literature which can be based, for example, on the use of series approximations (Lemos et al., 2023) or quadratures (Amaral, 2016), in the Post-Widder formula (Wang & Zhan, 2014), in the expansion of the original function into a series of orthogonal functions (Cohen, 2007), or even in the deformation of the Bromwich integration contour (Pilatti, 2019).

Within this context, this work consists of the application of the inverse Laplace Transform, numerically and analytically, in the resolution of a Parabolic Partial Differential Equation (Churchill, 1958). For the numerical solution, Weeks, Gaussian quadrature, and Power Series methods are used to invert numerically the LT. Weeks method, developed in 1965, uses a series expansion of the function f in terms of the Laguerre orthonormal functions (Cohen, 2007). The Gaussian Quadrature method approximates the definite integral on the interval $[-1, 1]$ of a function f by means of a weighted sum of the function values at specific points, determined so as to maximize the accuracy of the approximation. The Power Series method, on the other hand, is based on approximating the function in the Laplace domain by the function in the original domain, using a sum of exponential terms.

The accuracy of the methods is assessed by the Mean Absolute Error, calculated between the analytical and numerical profiles, which determines the optimal number of terms (N_o) that best fits the numerical approximation of the analytical one. Tests were performed for different values of t (0.005, 0.05, and 0.5) and x, with the results presented in tables and illustrative figures showing the scanning process for determining N_o and the behavior of the PDE solution.

The objective of this work is to compare the accuracy and stability of the Weeks, Gaussian Quadrature, and Power Series methods for numerical inversion of the Laplace Transform applied to the solution of a Parabolic Partial Differential equation, using the Mean Absolute Error to determine the optimal number of terms in the sum. The originality of this study lies in the systematic use of this metric to select N_0, providing a quantitative approach to assess the accuracy of the methods. Furthermore, the comparative analysis of the three algorithms applied to the same PDE enables a consistent evaluation of performance, offering valuable insights for selecting the most suitable method in similar problems. For a better understanding, this article is organized as follows: Section 2 describes the numerical inversion methods of LT and the numerical techniques used to solve the Parabolic Partial Differential Equation. Section 3 presents the equation to be solved, the equation after applying LT, and the analytical solution. Furthermore, the results obtained through simulations performed with the numerical inversion algorithms and the expression for calculating the Mean Absolute Error are presented. Section 4 discusses the efficiency of the methods and presents perspectives for future work.

2 NUMERICAL INVERSION METHODS

This section presents the mathematical formulation of the three numerical inversion methods of the Laplace Transform discussed in this work.

2.1 Weeks Method

The numerical inversion of the LT by Weeks method consists of approximating the function, in its initial domain, in terms of the transformed function, using the Laguerre polynomials as the basis of the expansion. According to Wang & Zhan (2014), this approximation is given by:

$$f_W(t) \approx e^{\sigma t} \sum_{j=0}^{N} a_j L_j \left(\frac{t}{g} \right), \tag{1}$$

where $L_j()$ represents the j-th Laguerre polynomial (Cohen, 2007), a_j is the Taylor coefficient. Still,

$$\sigma = \Psi - \frac{1}{2g}, \quad g = \tfrac{Tmax}{N}, \quad \Psi = \left(\alpha + \tfrac{1}{Tmax}\right) H\left(\alpha + \tfrac{1}{Tmax}\right),$$

where g is a scale factor, $H()$ is the Heaviside function, N is the number of terms adopted in the sum of the Laguerre polynomials and $Tmax$ is the highest value of t for the interval where the function is evaluated. In this study, $\alpha = 1$ is used following Wang & Zhan (2014), as this choice improves numerical stability for small t intervals. According to Weeks (1966), the convergence abscissa α can be obtained by analyzing $F(s)$ to locate the rightmost singularity of the transform. However, in practice, several authors, including Weeks (1966), Davies and Martin (1979), and Wang and Zhan (2014), recommend using α as an appropriate value to handle different classes of functions efficiently. This empirical choice simplifies implementation while maintaining robustness across a wide range of problems.

Furthermore, the Taylor coefficients are defined by

$$a_0 = \frac{1}{N+1} \sum_{k=0}^{N} h(\theta_k), \tag{2}$$

$$a_j = \frac{2}{N+1} \sum_{k=0}^{N} h(\theta_k) \cos(j\theta_k), \tag{3}$$

where

$$\theta_k = \frac{\pi}{2} \frac{2k+1}{N+1}, \tag{4}$$

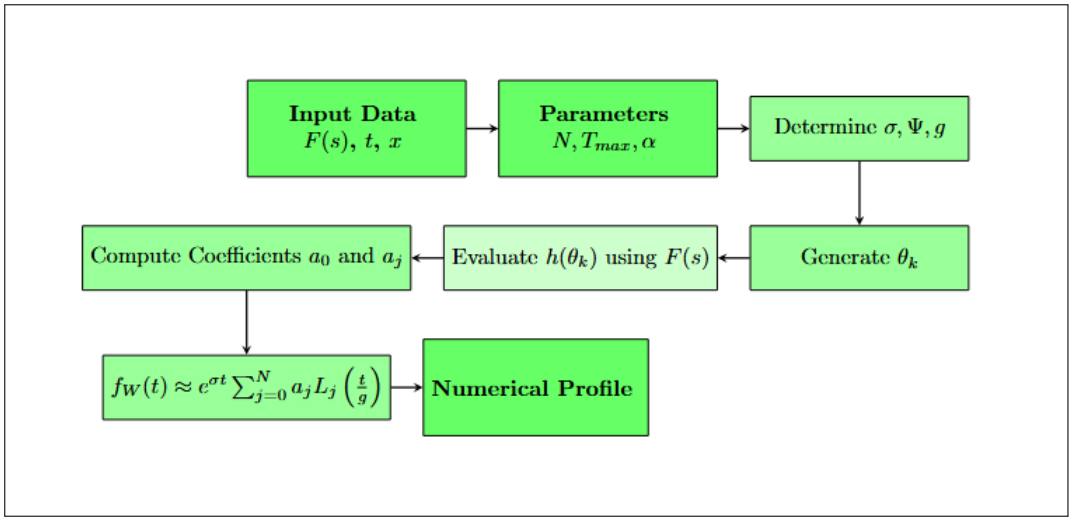
with $k = 0, 1, \ldots, N$,

$$h(\theta_k) = \frac{1}{T_n} \left\{ \Re \left[F\left(\Psi + \frac{i}{2g} \cot\left(\frac{\theta_k}{2} \right) \right) \right] - \cot\left(\frac{\theta_k}{2} \right) \Im \left[F\left(\Psi + \frac{i}{2g} \cot\left(\frac{\theta_k}{2} \right) \right) \right] \right\}, \tag{5}$$

for $\Re()$ and $\Im()$ representing, respectively, the real and imaginary parts of the function. In this work, it is assumed $T_n = 2g$.

Refer to Figure 1 for a systematic depiction of the Weeks Method algorithm, detailing the computational path from parameter input to final profile generation.

Figure 1 – Weeks Method Flowchart



Source: the authors (2025)

2.2 Gaussian Quadrature Method

Gaussian quadrature is a numerical method commonly used to calculate approximations of definite integrals, especially when they lack analytical solutions or are difficult to manipulate. The central idea is to approximate the integral as a

weighted sum of function values at specific points (called nodes or sampling points), chosen optimally to maximize accuracy.

In its general form, the Gaussian quadrature for an integral on the interval $[-1, 1]$ is given by:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i\,F(x_i),$$

where x_i are the quadrature nodes (roots of an orthogonal polynomial, such as Legendre's), and w_i are the associated weights.

Heydarian et al. (1981) brings an adaptation of the formula as follows:

$$f(t) \approx \sum_{i=1}^{N} a_i \frac{p_i}{t} F\left(\frac{p_i}{t}\right), \tag{6}$$

where p_i are the roots of the polynomial given by:

$$(-1)^N \sum_{r=0}^{N} a(N,r) p^{N-r} = 0, \tag{7}$$

with coefficients $a(N, r)$ defined as:

$$a(N,r) = \frac{(-1)^{N-r} N (N+r-1)!}{r!(N-r)!}. \tag{8}$$

The weights indicated by a_i, in equation (6), are obtained by solving the system:

$$\sum_{i=1}^{N} a_i p_i^{-r} = \frac{1}{r!}, \quad 0 \le r \le N-1, \tag{9}$$

whose matrix form is given by:

$$\underbrace{\begin{pmatrix} p_1^0 & p_2^0 & \cdots & p_N^0 \\ p_1^{-1} & p_2^{-1} & \cdots & p_N^{-1} \\ p_1^{-2} & p_2^{-2} & \cdots & p_N^{-2} \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{-(N-1)} & p_2^{-(N-1)} & \cdots & p_N^{-(N-1)} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}}_{\mathbf{a}} = \underbrace{\begin{pmatrix} \frac{1}{0!} \\ \frac{1}{1!} \\ \frac{1}{2!} \\ \vdots \\ \frac{1}{(N-1)!} \end{pmatrix}}_{\mathbf{b}}, \tag{10}$$

where the matrix A contains the values of p_i^{-r} and the vector $\mathbf{a}$ is formed by the weights described in the method as a_i.

To obtain the solution to the linear system described in equation (10), singular value decomposition is used (Bisgard, 2021) which can be written in matrix form as:

$$A = U\,\Sigma\,V^T, \tag{11}$$

or even,

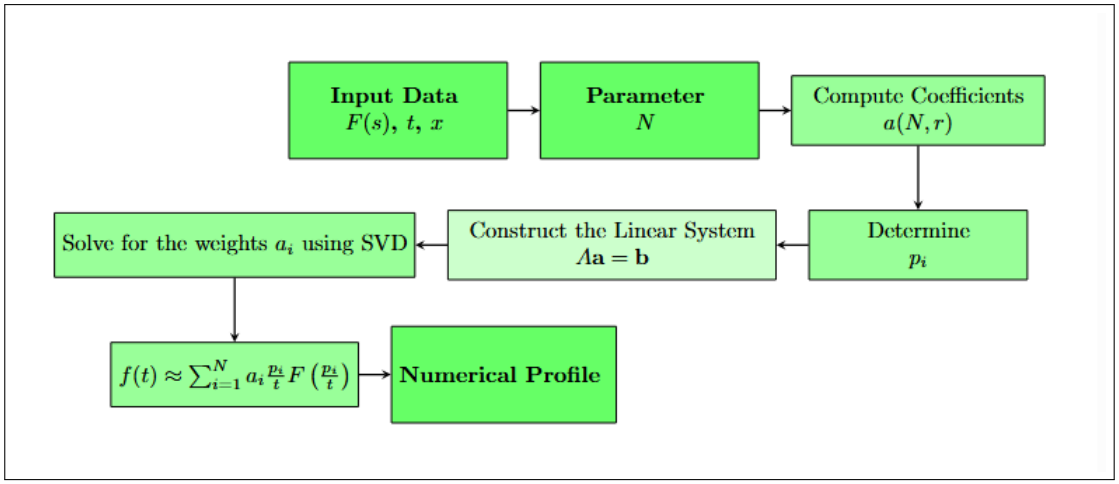
$$A = \underbrace{\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ u_{21} & u_{22} & \cdots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N1} & u_{N2} & \cdots & u_{NN} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1N} \\ v_{21} & v_{22} & \cdots & v_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N1} & v_{N2} & \cdots & v_{NN} \end{pmatrix}}_{V^T}, \tag{12}$$

where A is the matrix system, $U = [U_{m,i}]$ and $V = [V_{n,i}]$ are orthogonal matrices and Σ is a diagonal matrix containing the singular values σ_i. Thus, each element of the matrix A can be written as

$$A_{m,n} = \sum_{i=1}^{r} \sigma_i\, U_{m,i}\, V_{n,i}. \tag{13}$$

Figure 2 outlines the procedural steps for implementing the Gaussian Quadrature Method, highlighting the sequence from process formulation to obtaining the numerical profile.

Figure 2 – Gaussian Quadrature Flowchart



Source: the authors (2025)

2.3 Power Series Method

The numerical inversion method by Power Series is based on approximating the function f by a combination of exponentials of the type e^{-ikt}. Thus, the LT inversion problem requires determining the coefficients of these exponentials. The authors Chung & Sun (1986) e Cohen (2007) mathematically describe this process as follows:

$$f(t) = \sum_{i=0}^{N} \alpha_i g_i(t), \tag{14}$$

where $g_i(t) = e^{-ikt}$, k is an experimental parameter and the coefficients α_i, with $i = 0, 1, \ldots, N$, are the constants that will be determined during the process. This is equivalent to writing

$$f(t) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^N, \tag{15}$$

where $z = g_i(t)$. Since the Laplace Transform is given by

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt, \tag{16}$$

it is possible to obtain

$$F((j+1)k) = \int_0^{\infty} f(t) e^{-tk} g_j(t)\, dt. \tag{17}$$

Finally, multiplying equation (14) by $g_j(t)e^{-kt}$, integrating with respect to t from 0 to ∞, and substituting into equation (17), the obtained relation is

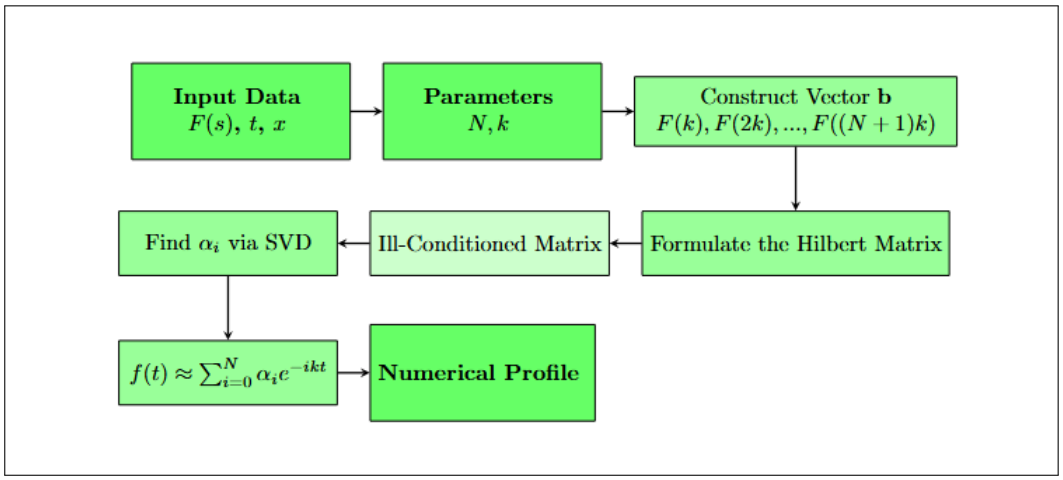
$$\sum_{i=0}^{N} \alpha_i \frac{1}{i+j+1} = F((j+1)k). \tag{18}$$

Note that equation (18) is a system of equations for $j = 0, 1, \ldots, N$, which can be represented in matrix form as:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N+1} & \frac{1}{N+2} & \frac{1}{N+3} & \cdots & \frac{1}{2N+1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} F(k) \\ F(2k) \\ \vdots \\ F((N+1)k) \end{bmatrix}. \tag{19}$$

One of the main challenges of this method is related to the matrix presented in equation (19), which corresponds to the Hilbert matrix. This matrix is known to be ill-conditioned, which worsens as the parameter N increases. To overcome this problem, the singular value decomposition (SVD) method is used as a stable way to solve the system. Figure 3 visually maps the operational flow of the Power Series Method, illustrating the necessary steps for the calculation of the numerical approximation.

Figure 3 – Power Series Flowchart



Source: the authors (2025)

3 RESULTS AND DISCUSSIONS

This section presents the results obtained from the use of the numerical inversion methods presented here in solving the PPDE proposed by Heydarian et al. (1981) using the Laplace Transform method. The computational algorithms were implemented in the free software OCTAVE, version 9.1.0, on an Intel(R) Core(TM) I3-10110U notebook, @2.10GHz, with a 64-bit Windows operating system and 4GB of RAM.

3.1 Parabolic Partial Differential Equation

The mathematical model in this investigation is described by

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}, \tag{20}$$

with initial and boundary conditions given by

$$u(x,0) = 1 \tag{21}$$

and

$$u(0,t) = 0, \quad u(1,t) = 0. \tag{22}$$

Applying LT to the variable t in equation (20) and to the boundary conditions, equation (22), and substituting the initial condition (equation (21)), the problem is rewritten in the Laplace domain and is now represented by

$$\frac{d^2\overline{U}(x,s)}{dx^2} - S\overline{U}(x,s) = -1, \tag{23}$$

and

$$\overline{U}(0,s) = 0, \quad \overline{U}(1,s) = 0. \tag{24}$$

The solution to the transformed problem, represented by the equations (23) and (24), is given by the following expression:

$$\overline{U}(x,s) = -\frac{1}{s}\cosh(\sqrt{s}x) + \frac{\cosh(\sqrt{s}) - 1}{s\ \sinh(\sqrt{s})}\ \sinh(\sqrt{s}x) + \frac{1}{s}. \tag{25}$$

After algebraic manipulations and substituting the addition formulas for hyperbolic functions (Spiegel & Liu, 2004) into the equation (25), this leads to

$$\overline{U}(x,s) = \frac{1}{s\ \sinh(\sqrt{s})}\left[\sinh[\sqrt{s}(x-1) - \sinh(\sqrt{s}x) + \sinh(\sqrt{s})\right]. \tag{26}$$

Applying the inverse Laplace Transform (Spiegel & Liu, 2004) and its properties, the equation (26), which represents the solution of the PPDE, is written in the domain of t as

$$u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)^2\pi^2 t}}{2n+1}\sin[(2n+1)\pi x]. \tag{27}$$

The analytical series was truncated at $M = 30$ terms to ensure accuracy across all tested values of t. This choice guarantees that the Mean Absolute Error remains below the order of 10^{-3} for all time values considered in this study. Therefore, $M = 30$ was chosen as a conservative value to guarantee precision for the entire range of t considered in this work.

3.2 Application of numerical inversion methods to obtain the solution of the PPDE

Since the LT inversion algorithms employed in this paper approximate the desired solution in terms of a series, determining the most appropriate number of terms to be added, it is crucial for the numerical method to achieve the best possible accuracy and this is a key aspect in evaluating its efficiency (Galdino, 1995). However, although this choice can be made in many ways, the literature provides little information on the subject (Galdino, 1995).

This work proposes the Mean Absolute Error ($\overline{E}_{abs}$) as the metric for selecting the optimal N (or N_o). This strategy was first described by Calixto et al. (2021), also adopted by Paz et al. (2023), and is adapted here and written as,

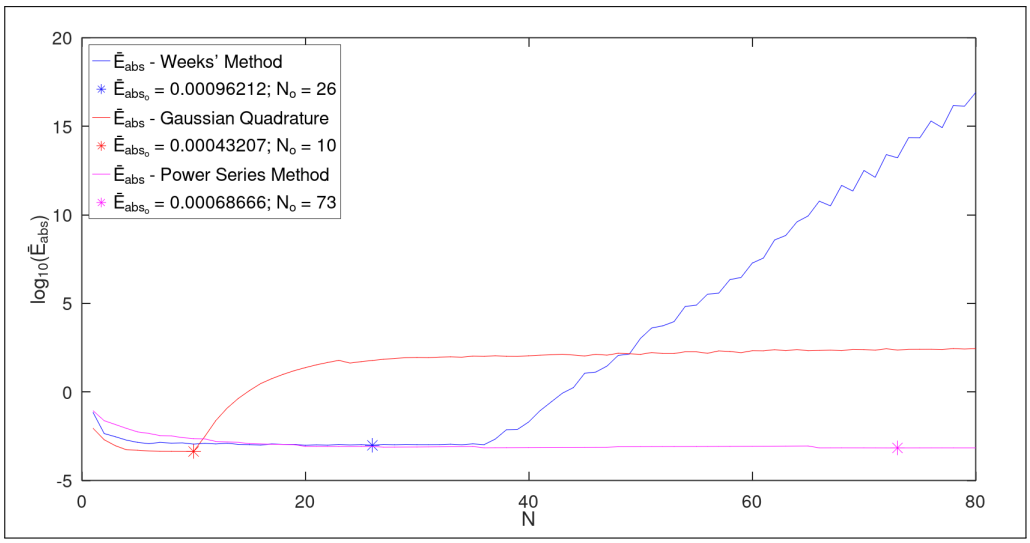
$$\overline{E}_{abs}(t) = \frac{1}{N_x}\sum_{i=1}^{N_x}|u_a(x_i,t) - u_n(x_i,t)|, x_i \in [0,1], t \in \Re^+, \tag{28}$$

where u_a represents the inverse transformed function obtained analytically, u_n the corresponding inversion obtained numerically and N_x the number of values of x used to obtain the solution profile. In particular, to generate the results, an interval $0 \leq x \leq 1$ with 11 subdivisions was chosen.

To evaluate the sensitivity of approximation of the PPDE solution by the proposed numerical methods in terms of the variable t, the Mean Absolute Error metric was applied as follows: for each desired instant of t (0.005, 0.05 and 0.5) and for each one of the numerical inversion methods of LT, profiles u_n were calculated for different values of N (1,..., 80) through a parameter scan and, from the equation (28), the $\overline{E}_{abs}$ was estimated. In this sequence of simulations, the value of N that resulted in the lowest $\overline{E}_{abs}$ was called the optimal N (or N_o) and this was chosen to generate the profiles presented here.

For the first case, where $t = 0.005$, when using the Gaussian Quadrature Method, a Mean Absolute Error of 4.3207×10^{-4} was obtained. For the Power Series and Weeks Methods, the errors were, respectively, 6.8666×10^{-4} and 9.6212×10^{-4} (Figure 4). Regardless of the computational effort (whether N_o is high or low), the results obtained present an error on the order of 10^{-4}.

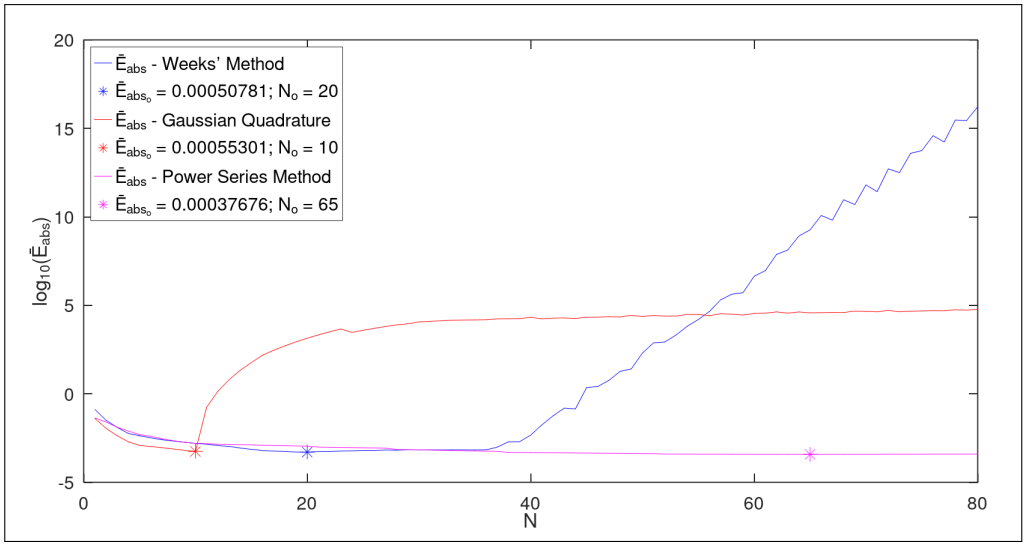
Figure 4 – Mean Absolute Error and N_0 at $t = 0.005$



Source: the authors (2025)

Figure 5 shows the Mean Absolute Errors for the proposed numerical inversion methods, applied to PPDE with $t = 0.05$. The Power Series Method presents a Mean Absolute Error of 3.7676×10^{-4} and the Weeks Method 5.0781×10^{-4}. The Gaussian Quadrature Method shows a Mean Absolute Error of 5.5301×10^{-4}.

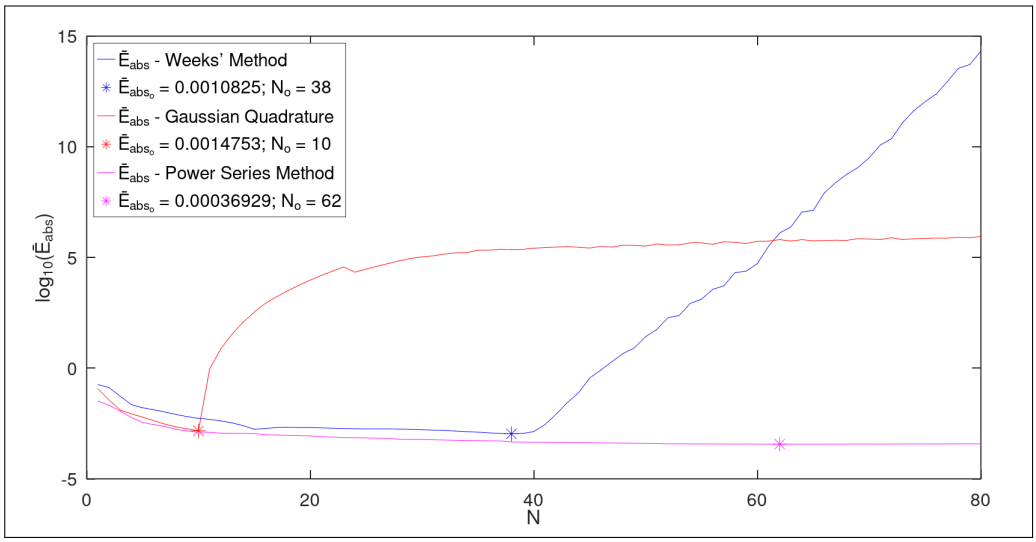
Figure 5 – Mean Absolute Error and N_0 at $t = 0.05$



Source: the authors (2025)

Figure 6 presents the last case studied here, where $t = 0.5$. For this situation, the Mean Absolute Error in the Power Series Method is smaller than for the other cases, being in the range of 10^{-4}, which is equal to 3.6929×10^{-4}. For the Weeks and Gaussian Quadrature methods, the Mean Absolute Errors are in the range of 10^{-3}, with their values, respectively, 1.0825×10^{-3} and 1.4753×10^{-3}.

Figure 6 – Mean Absolute Error and N_0 at $t = 0.5$



Source: the authors (2025)

It can also be highlighted that, in both cases, regardless of the values of t, the Gaussian Quadrature method present the lowest values of N_o among the methods studied.

In Table 1, the values obtained for the Mean Absolute Error and the respective optimal N are presented for each of the established values of t. The acronyms refer to the names of the methods: Weeks Method (WM), Gaussian Quadrature (GQ) and Power Series (PS). The results obtained are satisfactory and are in agreement with results described in the literature (Heydarian et al., 1981).

Table 1 – Mean Absolute Error versus Number of Terms N_o and Time t

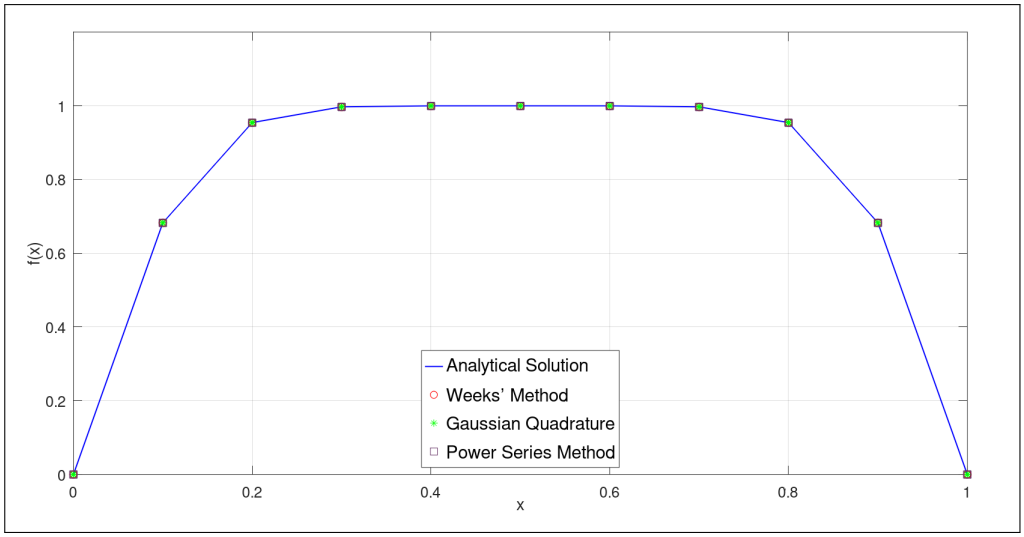
t	$N_o(MW)$	$N_o(GQ)$	$N_o(PS)$	$\overline{E}_{abs}(MW)$	$\overline{E}_{abs}(GQ)$	$\overline{E}_{abs}(PS)$
0.005	26	10	73	9.6212×10^{-4}	4.3207×10^{-4}	6.8666×10^{-4}
0.05	20	10	65	5.0781×10^{-4}	5.5301×10^{-4}	3.7676×10^{-4}
0.5	38	10	62	1.0825×10^{-3}	1.4753×10^{-3}	3.6929×10^{-4}

Source: the authors (2025)

By using the values of N_o obtained through scanning (Figures 4, 5 and 6) it was possible to draw the numerical profile that best fits the analytical one for the solution of the PPDE. These profiles were obtained by fixing the values for the variable t (0.005, 0.05 and 0.5) and evaluating the solution on the interval of x, $0 \leq x \leq 1$.

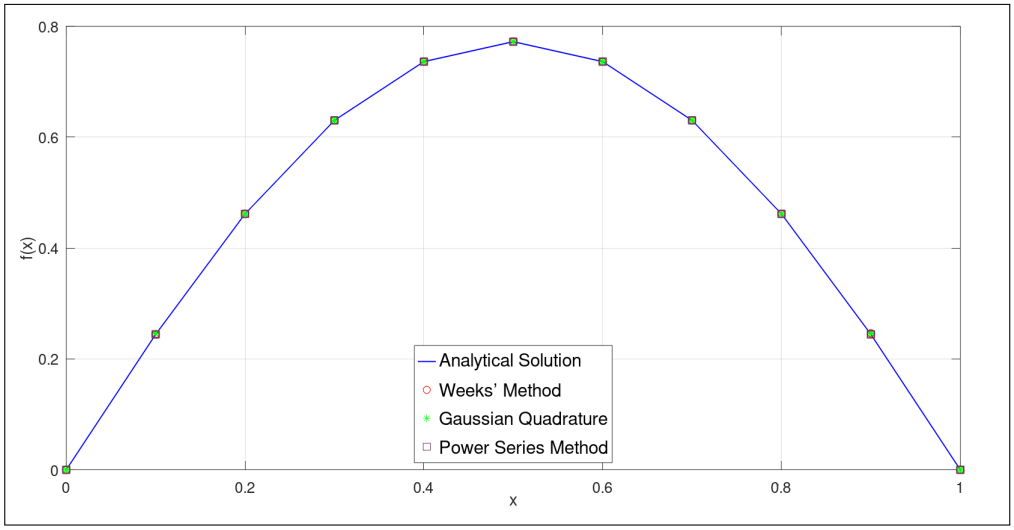
Figures 7, 8 and 9 show a comparison between the analytical curve and those obtained by Weeks, Gaussian Quadrature and Power Series numerical inversion methods with $t = 0.005$, $t = 0.05$ and $t = 0.5$, respectively. In particular, attention is drawn to Figure 9, which illustrates the difficulty of the Gaussian Quadrature (mainly) and Weeks methods in fitting the analytical solution for $t = 0.5$, indicating the sensitivity of these methods to variations in the value of t. Furthermore, in Figure 9, $N = 9$ (and not $N = N_o$) was adopted for generating the graph using the Gaussian Quadrature method. This change was necessary because the resulting profile using $N = N_o$ showed some negative values, which is incompatible with the behavior of the parabolic PDE. Therefore, $N = 9$ was selected as the smallest positive integer that preserves the coherence of the method and ensures the consistency of the results.

Figure 7 – Numerical vs. Analytical Solution Profile with $t = 0.005$



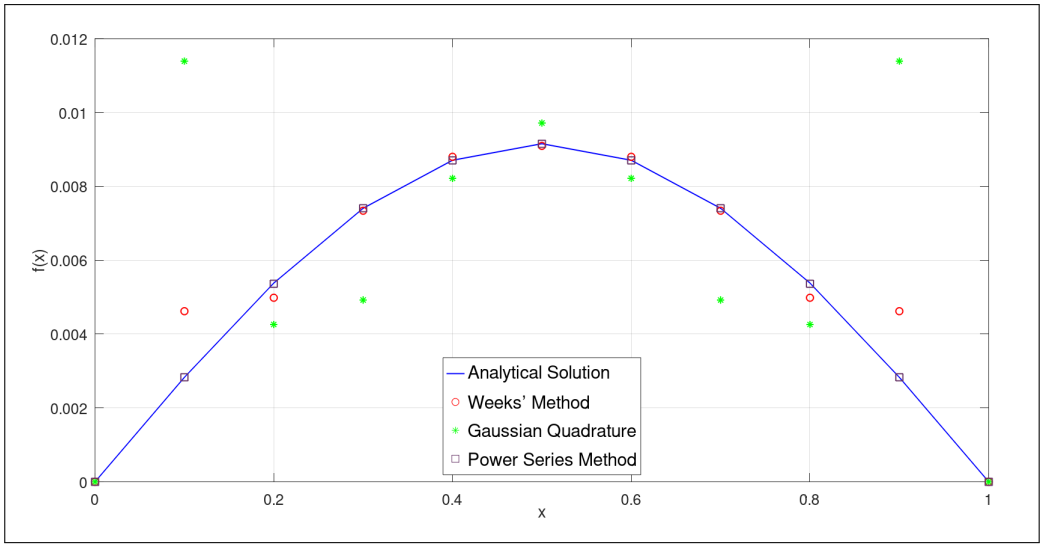
Source: the authors (2025)

Figure 8 – Numerical vs. Analytical Solution Profile with $t = 0.05$



Source: the authors (2025)

Figure 9 – Numerical vs. Analytical Solution Profile with $t = 0.5$



Source: the authors (2025)

In addition to the Mean Absolute Error graphs and the analytical and numerical profiles, this work also provides Tables 2, 3 and 4, which present comparisons between the analytical and numerical solutions for different values of x and t, in order to allow a more punctual evaluation of the results. The numerical values provided here were generated by using the N_o found for each method at each instant of t.

Table 2 – Analytical and Numerical Solutions for $t = 0.005$

x	0.1	0.2	0.3	0.4	0.5
Analytical Solution	0.682689492	0.954499736	0.997300204	0.999936655	0.999998853
Weeks Method	0.682572242	0.954237738	0.996879738	0.999390656	0.999405261
Gaussian Quadrature	0.682743752	0.954500040	0.997300220	0.999936656	0.999998853
Power Series	0.682507171	0.954231633	0.996938827	0.999512586	0.999552859

Source: the authors (2025)

Table 3 – Analytical and Numerical Solutions for $t = 0.05$

x	0.1	0.2	0.3	0.4	0.5
Analytical Solution	0.244248060	0.461646521	0.630401071	0.736327185	0.772311606
Weeks Method	0.242644662	0.461664786	0.630422937	0.736353628	0.772339244
Gaussian Quadrature	0.246046946	0.461337447	0.630479212	0.736311965	0.772312546
Power Series	0.244231355	0.461667902	0.630395067	0.736327048	0.772309660

Source: the authors (2025)

Table 4 – Analytical and Numerical Solutions for $t = 0.5$

x	0.1	0.2	0.3	0.4	0.5
Analytical Solution	0.002829665	0.005382343	0.007408160	0.008708815	0.009156990
Weeks Method	0.004620638	0.004985196	0.007342433	0.008802172	0.009097815
Gaussian Quadrature	0.011392618	0.004260839	0.004922602	0.008220172	0.009717331
Power Series	0.002831595	0.005363401	0.007409449	0.008704981	0.009154214

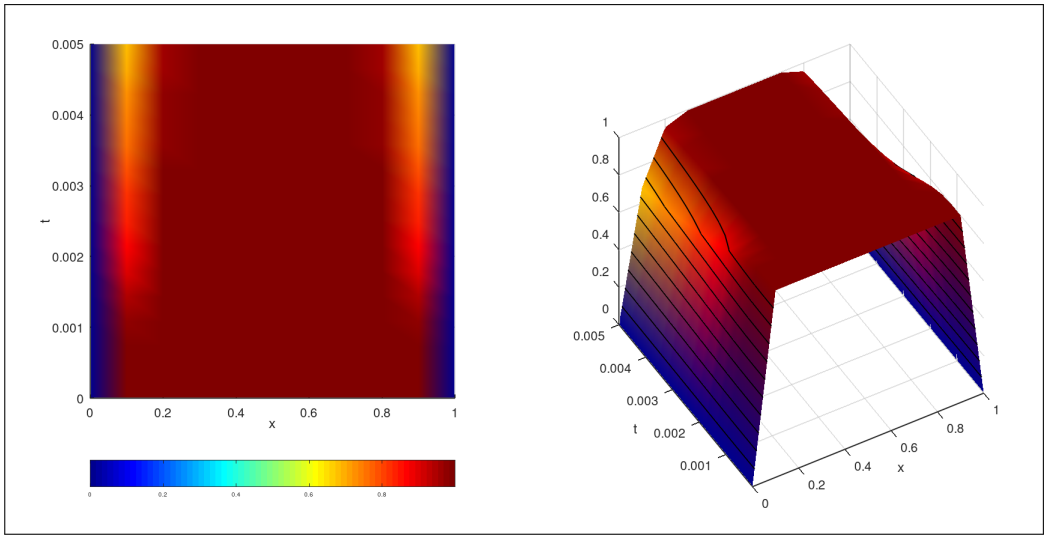
Source: the authors (2025)

For illustrative purposes of the behavior of the modeled phenomenon, surfaces were generated to represent the PPDS solution. In Table 4, $N = 9$ was adopted for the Gaussian Quadrature method according to the same justification presented for Figure 9.

Figures 10, 11 and 12 present the numerical solutions obtained for the PPDE for the intervals $0 \leq t \leq 0.005$, $0 \leq t \leq 0.05$ and $0 \leq t \leq 0.5$, respectively. Although the solution surfaces are visually identical regardless of the numerical method chosen, Gaussian Quadrature method was used because it provides results comparable to those obtained analytically with the lowest N among the three techniques studied here.

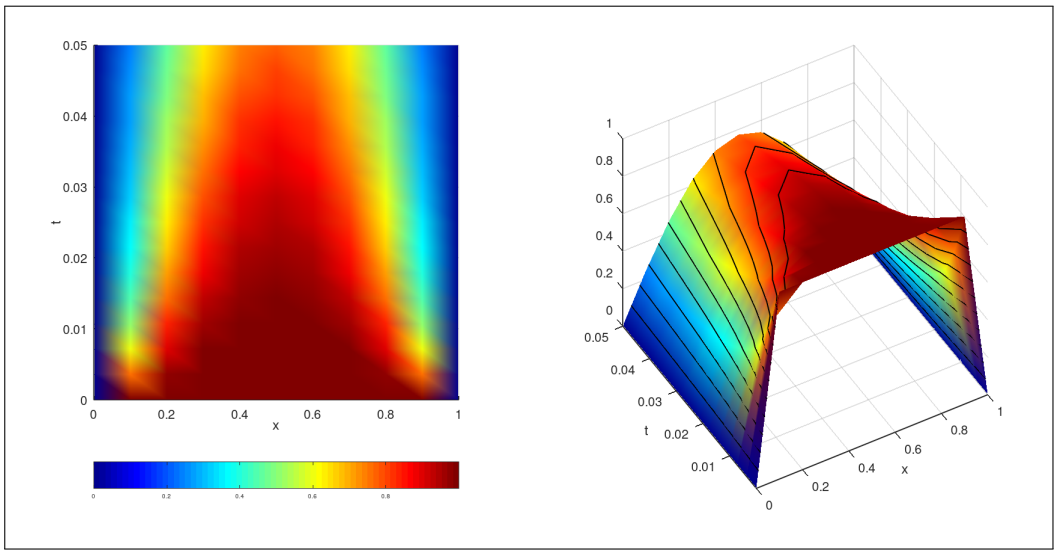
It is worth noting that the surfaces were generated within these three t ranges in an attempt to circumvent the scaling problems encountered due to the rapid decay of the PPDE solution. Furthermore, for the generation of Figure 12, $N = 9$ was used (and not $N = N_o$) for the same reasons discussed previously.

Figure 10 – Gaussian Quadrature with $t = 0.005$



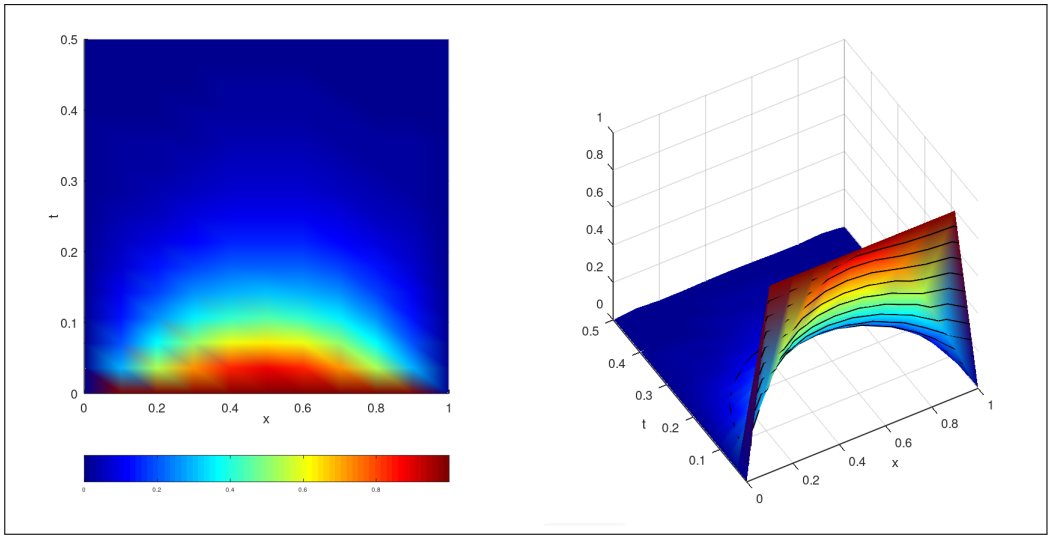
Source: the authors (2025)

Figure 11 – Gaussian Quadrature with $t = 0.05$



Source: the authors (2025)

Figure 12 – Gaussian Quadrature with $t = 0.5$



Source: the authors (2025)

4 FINAL CONSIDERATIONS

In this work, the Weeks, Gaussian Quadrature and Power Series methods were evaluated as alternatives for numerical inversion of the Laplace Transform when applied to obtain the solution of a Parabolic Partial Differential Equation. In the process, it was understood that the calculation of the Mean Absolute Error was assertive as a way to determine the value of N_o (the value of N responsible for truncating the number of terms of the series used by each inversion method, and which provides the best fit of the numerical profiles compared to the analytical ones). It is noteworthy that the value of N_o was obtained by scanning for each of the chosen values of t.

Observing the results, it was found that as t increases, N_o also changes (with the exception of the Gaussian Quadrature method). It was noticed that the Mean Absolute Errors obtained by the Weeks and Gaussian Quadrature methods vary significantly depending on the value of t used, ranging from 10^{-3} to 10^{-4}. This sensitivity was not observed for the Power Series method, whose Mean Absolute Errors remained on the order of 10^{-4}. This emphasizes the strong dependence of numerical stability and convergence properties on the time parameter t, highlighting its critical influence on the robustness and accuracy of the PPDE solution within the studied class of problems.

For large values of t, the analytical solution of the PPDE decays rapidly toward zero, making the numerical approximation highly sensitive to rounding errors. In the Weeks method, the term $e^{\sigma t}$ can grow significantly if σ is not properly adjusted, amplifying small coefficient errors and causing oscillations. Similarly, Gaussian Quadrature suffers from numerical instability because the integrand involves e^{st}, which becomes very large for high values of t, leading to ill-conditioning in the computation of weights. These effects explain the divergence observed in the graphs for $t = 0.5$. To address this issue, parameter recalibration is recommended. However, it should be noted that proper calibration of these parameters is not the objective of this work.

The performance of numerical inversion methods of the Laplace Transform is influenced by their intrinsic parameters. In the Weeks method, the parameters σ, g, and α control the scaling and convergence of the series. Although α can be computed by analyzing the singularity structure of $F(s)$, several authors recommend adopting $\alpha = 1$ as a robust empirical choice for a wide range of functions. In this work, $\alpha = 1$ was adopted according to these recommendations, simplifying implementation while maintaining numerical stability. Meanwhile, σ and g were calculated based on T_{max} to guarantee convergence and maintain accuracy within the tested time intervals. In the Power Series method, the parameter k defines the spacing of the exponential terms and was empirically adjusted to minimize the Mean Absolute Error. In this work, k was selected within the interval $[0.05, 0.1]$ for each one profiles. In Gaussian Quadrature, the roots and weights depend on N and are obtained from Legendre polynomials; their distribution directly affects the conditioning of the system. Improper selection of these parameters can lead to oscillations or divergence, especially for large values of t, as previously discussed. Although parameter recalibration could improve robustness, it should be noted that the optimization of these parameters is not the objective of this work.

It's worth noting that all the algorithms used in this work were implemented by the group in Octave, a freely available software with a programming language accessible to all academic levels. Overall, these programs were able to provide good results, very close to those obtained analytically, at an extremely low computational cost.

As a proposal for future work, it is suggested that the methods described in this study be applied to the solution of different classes of PPDEs, involving functions with special features such as source terms, boundary or initial conditions, or even analytical solutions. Furthermore, it is intended to investigate the influence of the problem's intrinsic parameters on the efficiency of the methods.

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