Solutions of the atmospheric advection-diffusion equation

Tiziano Tirabassi

Institute ISAC of CNR, Bologna, Italy
e-mail: t.tirabassi @ isac.cnr.it

Abstract

Some recent results about solutions of the atmospheric advection-diffusion equation will be presented. Moreover, applications of the solutions to the study of dispersion phenomena will be suggested.

Resumo

Alguns recentes resultados sobre soluções da equação de difusão-advecção atmosférica serão apresentados. Além disso, serão sugeridas aplicações das soluções para o estudo de   fenômenos de dispersão

Introduction

Many solutions of advection-diffusion equation were proposed in literature (Tirabassi, 2003), although, mostly for stationary conditions and by making strong assumptions about the eddy diffusivity coefficients and wind speed profiles. In the last 2 years new solutions were found with any limitations to the functional form of wind and eddy diffusion vertical profiles. This method is known as GILTT approach (Generalized Integral Laplace Transform Technique). For more information about the solutions see Moreira et al (2005) and Moreira et al. (2006).

Analytical solutions can be useful to study the phenomenon of dispersion in atmosphere and, moreover, in evaluating the performances of numerical method (that solve numerically the advection diffusion equation) that could compare their results, not only against experimental data but, in an easy way, with the solution itself in order to check numerical errors.

Moments of the vertical concentration distribution

Based on the analytical expression of the concentration distribution it is possible to analytically express the statistical descriptors moments characterizing the vertical concentration distribution. These are defined as follows:

$$\mu_1 = \frac{\int_0^h zC(x,z)dz}{\int_0^h C(x,z)dz} \quad \text{and the remaining} \quad \mu_m = \frac{\int_0^h (z-\mu_1)^m C(x,z)dz}{\int_0^h C(x,z)dz} \; ; \; m = 2,3,4.$$

After substituting the solution: $C(x,z) = \sum_{i=0}^{\infty} \bar{c}_i(x)\psi_i(z)$, where

$\bar{c}_i(x)$ (x is the distance from the source) and $\psi_i(z)$ (z is the vertical) are the solutions of the transformed equation and the Sturm-Liouville problem respectively (for more details see the works of Moreira et al., 2005), we obtain the explicit expressions for the four moments. The first and the second moment are expressed in a dimensional form as:

$$\mu_1 = \frac{h}{\bar{c}_0(x)\pi^2} \sum_{i=1}^n \frac{\bar{c}_i(x)}{i^2}(\cos i\pi - 1) + \frac{h}{2}, \quad \mu_2 = \frac{2h^2}{\bar{c}_0(x)\pi^2} \sum_{i=1}^n \frac{\bar{c}_i(x)}{i^2}\cos i\pi + \frac{h^2}{3} - \bar{z}^2$$

where h is the height of the planetary boundary layer

After assigning $\bar{z} = \mu_1$ and $\sigma_z^2 = \mu_2$, it is convenient to express the remaining moments μ_3 and μ_4 in dimensionless form after defining the skewness and the kurtosis as: $Sk = \dfrac{\mu_3}{\sigma^3}, \quad Ku = \dfrac{\mu_4}{\sigma^4}$.

Aware of these dimensionless definitions we have the expressions

$$Sk = \frac{3}{\bar{c}_0(x)\pi^2}\left(\frac{h}{\sigma_z}\right)^3 \sum_{i=1}^n \frac{\bar{c}_i(x)}{i^2}\left[\left(1 - \frac{2}{i^2\pi^2}\right)\cos(i\pi) + \frac{2}{i^2\pi^2}\right] + \frac{1}{4}\left(\frac{h}{\sigma_z}\right)^3 - \frac{3\bar{z}}{\sigma_z} - \left(\frac{\bar{z}}{\sigma_z}\right)^3$$

$$Ku = \frac{4}{\bar{c}_0(x)\pi^2}\left(\frac{h}{\sigma_z}\right)^4 \sum_{i=1}^n \frac{\bar{c}_i(x)}{i^2}\left(1 - \frac{6}{i^2\pi^2}\right)\cos(i\pi) + \frac{1}{5}\left(\frac{h}{\sigma_z}\right)^4 - \frac{4\bar{z}Sk}{\sigma_z} - 6\left(\frac{\bar{z}}{\sigma_z}\right)^2 - \left(\frac{\bar{z}}{\sigma_z}\right)^4$$

An example of the four moments, using a power low wind profile and the Kz profile suggested in Degrazia et al. (2000), are presented in Figure 1

For each plume source height the centroid starts from the respective source height and slowly approaches to the asymptotic value of $0.5h$. The plots referring to the σ_z/h tends to the asymptotic value $\approx 0.3h$.

The highest source emissions show a nearly Gaussian distribution, this is particularly manifest in the S_k and the *Ku* plots already for short distances. The kurtosis plots are strongly depending on the source heights.

In all plot it is anyway very easy to highlight the terrain influence on the lowest sources. Kurtosis plots reveal a poorly peaked vertical distribution tendency in respect of the traditional Gaussian behaviour, which is still widely used for operative purposes.

The power law vertical behaviour of the horizontal wind profile ensures the wind to increase rapidly in the lower part of the ABL. For the convective cases, above the height $z \approx 0.2h$ the wind remains nearly constant. Because of such behaviour the highest sources releases seems to be poorly affected by the vertical structure of the horizontal wind. Furthermore they reach earlier than lower source release cases the tendency to approach to the asymptotic limit.
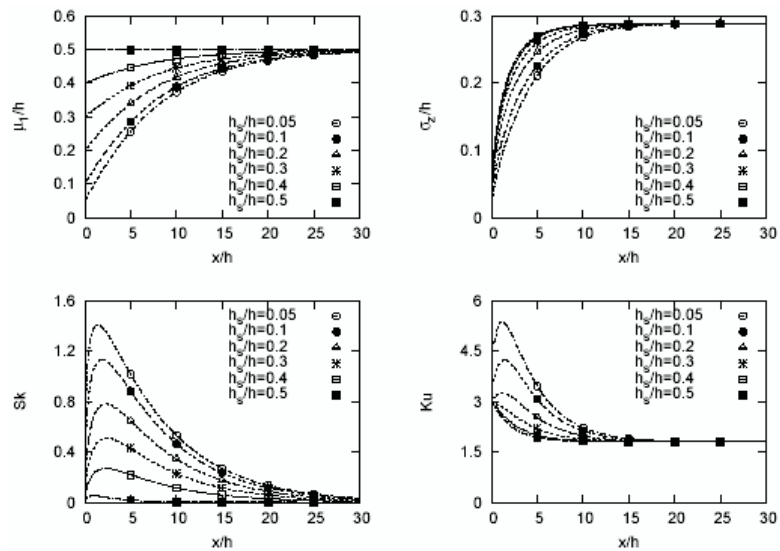

Figure 1. Moments for the concentration vertical distribution, for 6 source heights (hs). Monin-Obukhov length L = -30 m and wind velocity u = 4 ms^{-1}.

References

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