CIÊNCIA e NATURA

ISSN 2179-460X
OPEN ACCESS

Ci. e Nat., Santa Maria, v. 47, spe. 1, e90573, 2025• https://doi.org/10.5902/2179460X90573
Submitted: 01/10/25 • Approved: 01/10/25 • Published: 02/14/25

IV Jornada de Matematica e Matematica aplicada UFSM

Study of the average concentration of contaminants in a landfill

Estudo da concentração média de contaminantes em um aterro sanitário

Josiane Konradt [I], **Igor da Cunha Furtado** [II], **Régis Sperotto de Quadros** [I], **Daniela Buske** [I], **Guilherme Jahnecke Weymar** [I]

[I]Universidade Federal de Pelotas, RS, Brazil
[II]Instituto Federal Sul-rio-grandense, RS, Brazil

ABSTRACT

Groundwater plays an essential role in maintaining life and the environment, and its adequate management is crucial to ensuring the continuous availability of quality water for future generations. However, accidents that compromise its quality can happen, such as when leachate produced by Landfills penetrate the soil, reaching the water table and contaminating it. This study presents a two-dimensional model for the transport of contaminants in landfills and investigates the influence of the physical parameter Peclet number (Pe) on the average concentration of pollutants in a porous media. The model uses the Generalized Integral Laplace Transform Technique (GILTT) technique to obtain the analytical solution, considering the continuous and uniform leakage of pollutants from the municipal solid waste (MSW) storage cell. The simulation results show that, as the value of Pe increases, the average concentration of pollutants increases, affecting the water table more seriously.

Keywords: Sanitary landfills; Dispersion of pollutants; Retardation factor; GILTT; Mathematical modeling

RESUMO

As águas subterrâneas desempenham um papel essencial na manutenção da vida e do meio ambiente, sendo sua gestão adequada crucial para garantir a disponibilidade contínua de água de qualidade para as gerações futuras, entretanto acidentes que comprometam sua qualidade podem acontecer, como quando o chorume produzido por aterros sanitários penetram no solo, atingindo o lençol freático e contaminando-o. Este estudo apresenta um modelo bidimensional para o transporte de contaminantes em aterros sanitários e investiga a influência do parâmetro físico número de Peclet (Pe) na concentração média de poluentes meio poroso. O modelo usa a técnica *Generalized Integral Laplace Transform Technique* (GILTT) para obter a solução analítica, sendo considerado o vazamento contínuo e uniforme de poluente da célula de armazenamento de resíduos sólidos urbanos (RSU). Os resultados das simulações mostram que, à medida que o valor do Pe aumenta, a concentração média de poluentes cresce, afetando o lençol freático de forma mais grave.

Palavras-chave: Aterros sanitários; Dispersão de poluentes; Fator de retardamento; GILTT; Modelagem matemática

1 INTRODUCTION

Groundwater is a crucial part of Earth's hydrological system, playing a vital role in supplying fresh water to many communities around the world. Hirata *et. al* (2019), highlights that 52% of the 5,570 Brazilian municipalities depend totally (36%) or partially (16%) on groundwater for public supply.

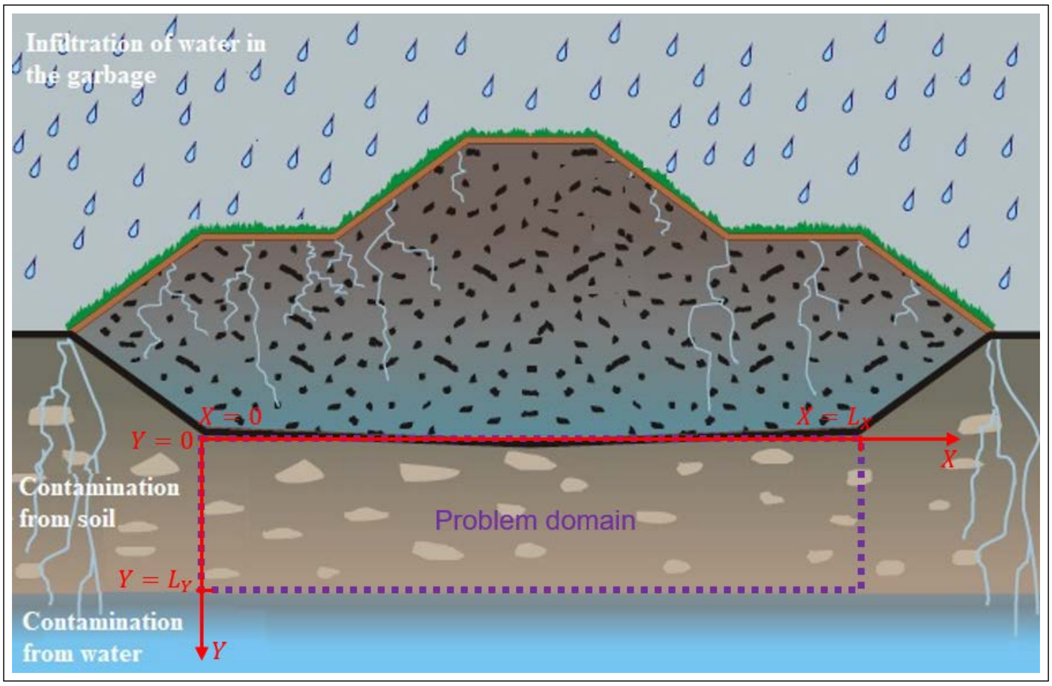
These underground water reserves are at risk of being severely polluted by harmful chemicals, industrial pollutants, pesticides and waste leaks from landfills.

The purpose of this work is to present the solution of a two-dimensional model of contaminant transport in a landfill and, through a simulation, verify the importance and influence of the physical parameter Peclet number in the study considered, observing the average concentration of pollutant in the porous medium. For this purpose, the dimensionless form of the effluent dispersion model will be used, considering the continuous and uniform leakage of pollutants from a municipal solid waste (MSW) storage cell. To solve the model, the Generalized Integral Laplace Transform Technique (GILTT) technique will be used, obtaining the solution of the analytical form of the transient mass transport model in a saturated porous medium.

2 METHODOLOGY

Figure 1 represents the two-dimensional model used in this work, depicting the simplified scheme of a solid waste storage cell. This is where the contaminant transport occurs through the porous medium until it reaches the groundwater table, where $Y = 0$ represents the horizontal boundary between the landfill and the soil, and $Y = L_Y$ represents the limit between the soil and the groundwater table.

Figure 1 – Cross-section of a landfill



Source: Adapted from CONDER (2017)

Equation (1), written in dimensionless form, describes the transport of pollutants in the saturated porous medium and can be found in the work of Albuquerque (2018):

$$R\frac{\partial C}{\partial \tau} = L^*\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} - Pe\frac{\partial C}{\partial Y}, \tag{1}$$

where R represents the soil retardation factor, C is the concentration of the contaminant in the liquid phase, τ represents time, L^* is the square of the X and Y ratio of dimensions $\left(L^* = (\frac{L_Y}{L_X})^2\right)$, and Pe is the Peclet number.

The initial condition of the problem is given by:

$$C(X, Y, 0) = C_0, \tag{2}$$

where C_0 is the initial concentration of the contaminant in the solid waste storage cell.

The boundary conditions in dimensionless form and in the X direction are given by:

$$\frac{\partial C}{\partial X}(0, Y, \tau) = 0, \qquad \frac{\partial C}{\partial X}(1, Y, \tau) = 0, \tag{3}$$

where zero flux conditions are used at the boundaries of the domain in X.

The boundary conditions in the Y direction are given by:

$$C(X,0,\tau)=1, \tag{4}$$

Equation (1) is subject to the interface condition (4), which corresponds to continuous and uniform leachate leakage in a solid waste storage cell, in addition to:

$$\frac{\partial C}{\partial Y}(X,1,\tau)+BiC(X,1,\tau)=0, \tag{5}$$

where Bi is the Biot number, and this condition represents the convective flow at the contact point between the soil and the groundwater table.

With the dimensionless equations represented by Eqs. (1) - (5), the resolution of the problem begins by applying the Superposition Method (Hahn & Özisik, 1993):

$$C(X,Y,\tau)=C^{*}(X,Y,\tau)+C_E(Y), \tag{6}$$

where C^{*} is an auxiliary function that carries the homogeneous boundary condition, and C_E is the solution for the steady-state problem. Thus, substituting equation (6) into the expression governing the transport of pollutants in the porous medium equation (1):

$$R\frac{\partial C^{*}}{\partial \tau}=L^{*}\frac{\partial^2 C^{*}}{\partial X^2}+\frac{\partial^2 C^{*}}{\partial Y^2}+\frac{d^2 C_E}{dY^2}-Pe\left(\frac{\partial C^{*}}{\partial Y}+\frac{dC_E}{dY}\right). \tag{7}$$

From equation (7), two differential equations are obtained, one, an ordinary differential equation (ODE) and, the other, a partial differential equation (PDE).

The ODE extracted from equation (7) is represented by the following expression:

$$\frac{d^2 C_E}{dY^2}-Pe\frac{dC_E}{dY}=0, \tag{8}$$

and the boundary conditions are given by the following equations:

$$C_E(0)=1, \qquad \frac{dC_E(1)}{dY}+BiC_E(1)=0. \tag{9}$$

Using the boundary conditions given by (9), the analytical solution of equation (8) is obtained:

$$C_E(Y) = \frac{e^{Pe}(Pe + Bi) - Bie^{PeY}}{e^{Pe}(Pe + Bi) - Bi}. \tag{10}$$

The PDE obtained from equation (7) is given by:

$$R\frac{\partial C^*}{\partial \tau} = L^*\frac{\partial^2 C^*}{\partial X^2} + \frac{\partial^2 C^*}{\partial Y^2} - Pe\frac{\partial C^*}{\partial Y}, \tag{11}$$

and the boundary conditions in the X direction are provided by the following equations:

$$\frac{\partial C^*}{\partial X}(0, Y, \tau) = 0, \qquad \frac{\partial C^*}{\partial X}(1, Y, \tau) = 0, \tag{12}$$

and the boundary conditions in the Y direction are given by:

$$C^*(X, 0, \tau) = 0, \qquad \frac{\partial C^*}{\partial Y}(X, 1, \tau) + BiC^*(X, 1, \tau) = 0, \tag{13}$$

and the initial condition is given by:

$$C^*(X, Y, 0) = C_0 - C_E(Y). \tag{14}$$

To obtain the solution $C^*(X, Y, \tau)$ of equation (11), the GILTT method is used. Firstly, the auxiliary Sturm-Liouville problem is considered in the X direction:

$$\frac{d^2\varphi}{dX^2} + \frac{\lambda^2}{L^*}\,\varphi = 0, \qquad \frac{d\varphi(0)}{dX} = 0, \qquad \frac{d\varphi(1)}{dX} = 0, \tag{15}$$

where the differential equation has as a solution the eigenfunctions (Hahn & Özisik, 1993):

$$\varphi_n(X) = \cos\left(\frac{\lambda_n}{\sqrt{L^*}}X\right), \tag{16}$$

applying the boundary conditions leads to the eigenvalues associated with each eigenfunction:

$$\lambda_n = n\pi\sqrt{L^*} \qquad\qquad n = 0, 1, 2, ... \tag{17}$$

In addition, the concentration is expanded as a series in terms of the eigenfunctions and then substituted into equation (11). By applying the integral operator $\int_0^1(.)\varphi_m(X)dX$ on both sides of the equation, noting that, by equation (15), $\varphi_n''(X) = -\lambda_n^2\varphi_n(X)$, an equation is obtained where all summations have zero terms, except when $m = n$. Therefore, the following expression is obtained:

$$R\frac{\partial \overline{C}_n(Y,\tau)}{\partial \tau} = -\lambda_n^2\overline{C}_n(Y,\tau) + \frac{\partial^2 \overline{C}_n(Y,\tau)}{\partial Y^2} - Pe\frac{\partial \overline{C}_n(Y,\tau)}{\partial Y}. \tag{18}$$

For the resolution of the PDE (18), the auxiliary Sturm-Liouville problem is first solved in Y:

$$\frac{d^2\psi}{dY^2} + \beta^2\psi = 0, \quad \psi(0) = 0, \quad \frac{d\psi(1)}{dY} + Bi\psi(1) = 0, \tag{19}$$

where the differential equation has as a solution the eigenfunctions (Hahn & Özisik, 1993):

$$\psi_k(Y) = \operatorname{sen}(\beta_k Y), \tag{20}$$

applying the boundary conditions leads to the eigenvalues associated with each eigenfunction, which must satisfy a transcendental equation, where the roots are calculated by the Newton-Raphson method.

Next, the solution of the PDE (18) is expanded as a series in terms of the eigenfunctions and then the substitution in the expansion in equation (18) is performed. Also, applying the integral operator $\int_0^1(.)\psi_l(Y)dY$ on both sides of the equation, noting that, by equation (19), $\psi_k''(Y) = -\beta_k^2\psi_k(Y)$ and regrouping the terms, the obtained equation can be rewritten in matrix form:

$$A \cdot Z'(\tau) + B \cdot Z(\tau) = 0, \tag{21}$$

where $Z(\tau) = \widetilde{C}k$, with $k = 0, 1, 2, ...$; $A = ak, l$, where $a_{k,l} = R\int_0^1 \psi_k(Y)\psi_l(Y)dY$; and $B = b_{k,l}$, where $b_{k,l} = (\lambda_n^2 + \beta_k^2)\int_0^1 \psi_k(Y)\psi_l(Y)dY + Pe\int_0^1 \psi_k'(Y)\psi_l(Y)dY$. Also, considering $F = A^{-1} \cdot B$, equation (21) is rewritten as:

$$Z'(\tau) + F \cdot Z(\tau) = 0. \tag{22}$$

The initial condition of the matrix differential equation, equation (22), is obtained by applying the same procedures performed in the PDE on equation (14), and thus, the initial condition is well defined.

To solve the matrix ODE (22), Laplace transform is applied on both sides. In this problem, it is assumed that the matrix F is diagonalizable, and $F = X \cdot D \cdot X^{-1}$ is written, where D is the diagonal matrix whose elements are the eigenvalues of F, X is the matrix whose columns constitute the linearly independent eigenvectors of F, and X^{-1} is its inverse. Therefore, it is concluded that the solution of the matrix ODE (22) is:

$$Z(\tau) = X \cdot G(\tau) \cdot X^{-1} \cdot Z(0). \tag{23}$$

Thus, the solution of the two-dimensional model of pollutant dispersion in porous media, represented by equations Eqs. (1) - (5), is given by:

$$C(X,Y,\tau) = \sum_{n=0}^{N} \varphi_n(X) \left[\sum_{k=0}^{K} \psi_k(Y) \widetilde{C}_k(\tau) \right] + C_E(Y), \tag{24}$$

where $\varphi_n(X)$ is defined by equation (16), $\psi_k(Y)$ is defined in equation (20), and $\widetilde{C}_k(\tau)$ is defined by (23).

3 RESULTS AND DISCUSSION

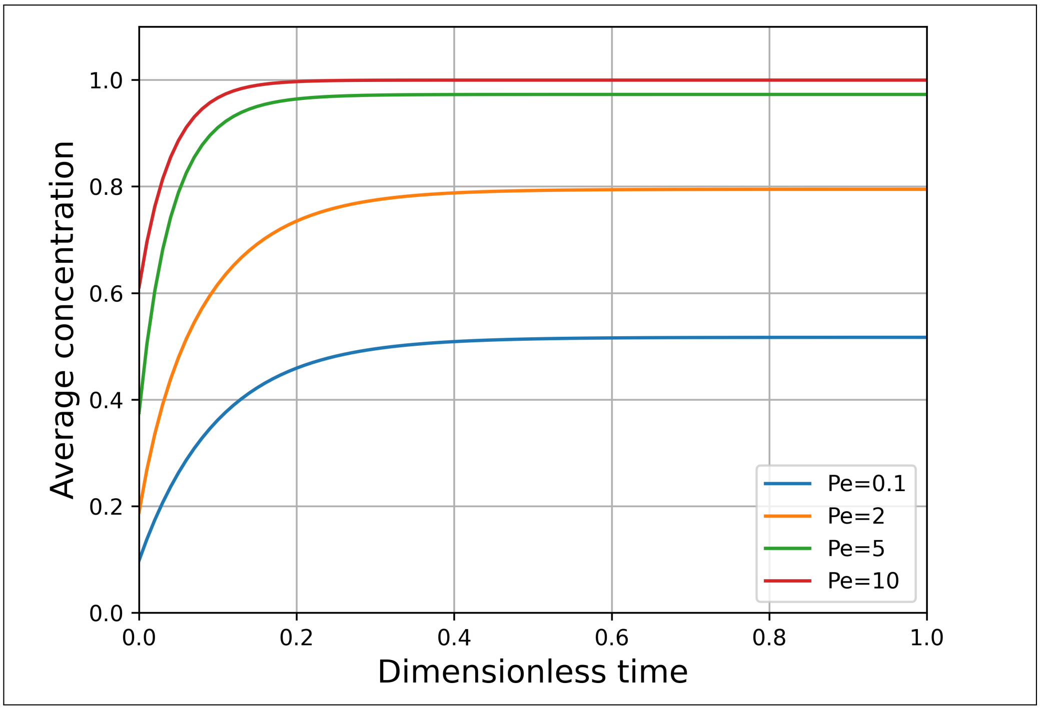
Based on the solution found, represented by equation (24), a study was carried out on the dimensionless average concentration for the contaminant concentration field in terms of dimensionless time through the online environment Google Colaboratory, in Python language, which is defined by:

$$C_m = \frac{\int_0^1 e^{-PeY} \cdot C(X,Y,\tau)}{\int_0^1 e^{-PeY} dY}, \tag{25}$$

where, for calculation purposes C_m was considered at the position of $X = 0.5$. Furthermore, it should be noted that, in the simulations, $Bi = 2000$ was used.

Firstly, the influence of the Pe parameter on the average concentration will be analyzed.

Figure 2 – Average concentration for $L^* = 1$ and $R = 1$



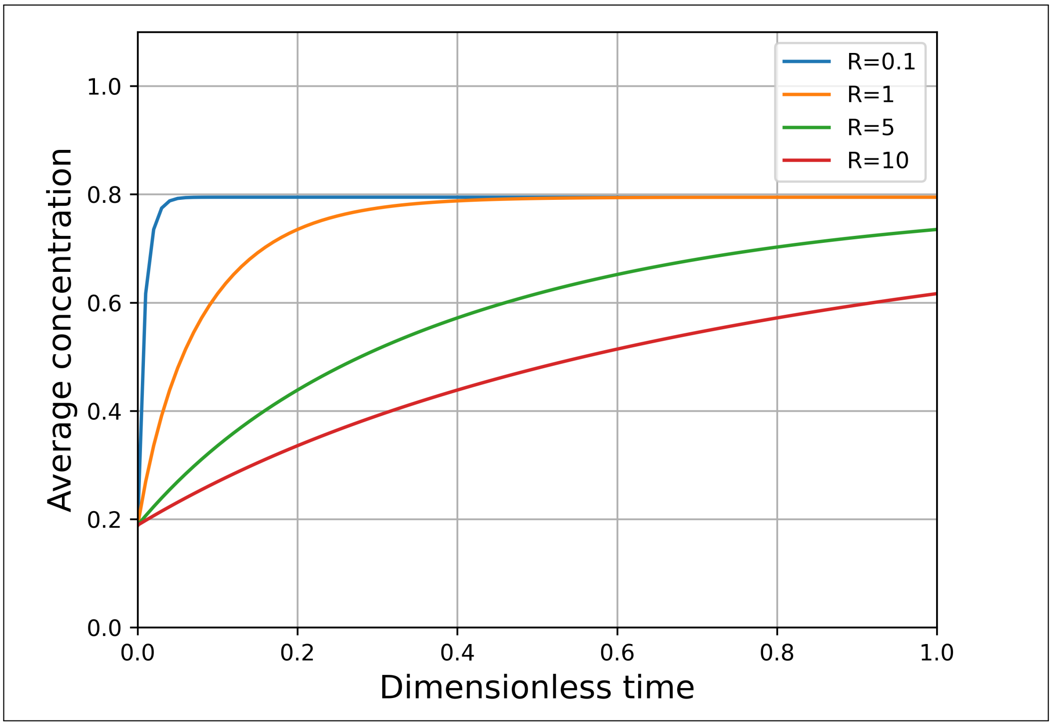
Source: Authors (2025)

With $Pe = 0.1$, it is observed that over time the pollutant flow can be considered slow, in addition to the fact that for $\tau > 0.50$ the distribution of the average concentration of the pollutant does not change. In other words, for the values of the Pe parameter considered to be low, the pollutant flow is slow and the average concentration reaches values below 0.6.p

However, increasing the value of the parameter to $Pe = 10$, it is clear that the pollutant originating from the MSW cell contaminates the soil more significantly in a shorter time, due to the fact that the contaminant flows at a greater speed and contaminates less the soil, that is, the advection process is greater than the diffusion process. Therefore, the possible causes of the groundwater can be significant, due to the concentration of pollutants that can reach it. Furthermore, it is noted that for $\tau = 0.20$ the average concentration reached the maximum value of 1, that is, it can be considered as a high index.

Now, the influence of the R parameter on the average concentration of contaminants in a landfill will be presented.

Figure 3 – Average concentration for $L^{*} = 1$ and $Pe = 2$



Source: Authors (2025)

When considering the parameter $R = 0.1$, it is clear that at a $\tau < 0.1$, the average concentration reaches a value of 0.8. However, considering a higher value of R, such as $R = 10$, the average concentration does not reach stability at $\tau = 1$, requiring a longer time, but it is clear that the concentration levels of contaminants are smaller. Therefore, through the simulations, it is possible to observe that as the value of R increases, the average concentration will be lower. This indicates that the contaminant flows at a greater speed, originating less soil contamination, that is, the contaminant concentration is retained by the soil porosity, thus requiring a longer time for the contaminant concentration to reach the water table.

4 CONCLUSIONS

With the present work it is possible to conclude that, depending on the value of the Pe parameter considered in the simulations, the average concentration of the contaminant changes, thus differing in the way in which the water table can be reached. Thus, when considering increasing values for the Pe parameter, the average concentration increases. This means that the contaminant moves more quickly through the soil, resulting in less contamination of this environment, but affecting water resources more substantially.

As the next steps in the research, we plan to investigate how other parameters influence the spread of contaminants and whether they more significantly affect the water table.

ACKNOWLEDGEMENTS

We thank the Coordination for the Improvement of Higher Education Personnel (CAPES) for funding our research.

REFERENCES

Albuquerque, F. A. (2018). *Estudo da propagação de contaminante em aterros sanitários via GITT*. [Tese, Programa de Pós-Graduação em Engenharia Mecânica, Universidade Federal da Paraíba]. Repositório Institucional da UFPB.

CONDER (2017). *Manual de Operação de Aterros Sanitários*. Web Archive.

Hahn, D. W. & Özisik, M. N. (1993). *Heat conduction*. Nova Jersey.

Hirata, R., Suhogusoff, A. V., Marcellini, S. S., Villar, P. C., & Marcellini, L. (2019). *A revolução silenciosa das águas subterrâneas no Brasil: uma análise da importância do recurso e os riscos pela falta de saneamento*. Instituto Trata Brasil.

Author contributions

1 – Josiane Konradt (Corresponding Author)
Mathematician
https://orcid.org/0000-0002-4689-8616 • josianekonradt@gmail.com
Contribution: Conceptualization, Data Analysis, Literature Review, Methodology, Writing – Original Draft Preparation

2 – Igor da Cunha Furtado
Mathematician
https://orcid.org/0000-0002-0776-3225 • igorjara@gmail.com
Contribution: Conceptualization, Data Analysis, Literature Review, Methodology, Writing – Original Draft Preparation

3 – Régis Sperotto de Quadros
Mathematician
https://orcid.org/0000-0002-9720-8013 • quadros99@gmail.com
Contribution: Methodology, Data Analysis, Writing – Review & Editing

4 – Daniela Buske
Mathematician
https://orcid.org/0000-0002-4573-9787 • danielabuske@gmail.com
Contribution: Conceptualization, Methodology, Data Analysis, Writing – Review & Editing

5 – Guilherme Jahnecke Weymar
Mathematician
https://orcid.org/0000-0001-8216-9122 • guilhermejahnecke@gmail.com
Contribution: Conceptualization, Data Analysis, Literature Review, Methodology, Writing – Original Draft Preparation

How to cite this article

Konradt, J., Furtado, I. da C., Quadros, R. S. de, Buske, D., & Weymar, G. J. (2025). Study of the average concentration of contaminants in a landfill. *Ciência e Natura*, Santa Maria, v. 47, spe. 1, e90573. DOI: https://doi.org/10.5902/2179460X90573.