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The asymptotic homogenization method applied to the elastostatic model of functionally graded microperiodic Euler-Bernoulli beams

O método de homogeneização assintótica aplicado ao modelo elastoestático de vigas de Euler-Bernoulli microperiódicas funcionalmente graduadas

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ABSTRACT

This work presents the application of the asymptotic homogenization method to a problem which models the mechanical equilibrium of functionally graded microperiodic Euler-Bernoulli beams clamped at both ends and subjected to a microperiodic distributed load. The five-term formal asymptotic solution is obtained in terms of the solution of the homogenized problem and the periodic solutions of the local problems, for whose existence a new result is presented. Analytical expressions for the homogenized and local solutions are provided. The exact solution of the problem, which is seldom available, is also provided for comparison purposes.

Keywords: Euler-Bernoulli beam; Asymptotic homogenization method; Formal asymptotic solution; Homogenized and local problems

RESUMO

Este trabalho apresenta a aplicação do método de homogeneização assintótica a um problema que modela o equilíbrio mecânico de vigas de Euler-Bernoulli microperiódicas funcionalmente graduadas fixadas em ambas as extremidades e submetidas a uma carga microperiódica distribuída. A solução assintótica formal de cinco termos é obtida em termos da solução do problema homogeneizado e das soluções periódicas dos problemas locais, para cuja existência é apresentado um novo resultado. São fornecidas expressões analíticas para as soluções homogeneizada e locais. A solução exata do problema, raramente disponível, também é fornecida para fins de comparação.

Palavras-chave: Viga de Euler Bernoulli; Método de homogeneização assintótica; Solução assintótica formal; Problemas homogeneizado e locais

1 INTRODUCTION

The Euler-Bernoulli beam theory (Rao, 2016) provides a classical fundamental model for analyzing the behavior of slender (i.e. ideally one-dimensional) structures under various loading conditions. However, when dealing with beams made of heterogeneous materials, the traditional analysis based on the classical Euler-Bernoulli beam theory becomes inadequate due to the complexity of the material properties. Moreover, when considering more accurate representations of the micro-heterogeneous structure of the materials real beams are made of, direct mathematical or computational treatment becomes particularly difficult, as the values of its properties change very rapidly with respect to position, which mathematically translates to differential equations with rapidly oscillating coefficients. This common feature of micro-heterogeneous materials is caused by the so-called separation of structural scales, that is, the heterogeneity occurs at the local scale (i.e. the microscale) whereas appearing homogeneous at the global scale (i.e. the macroscale). Such separation of scales is characterized by the geometric parameter ε, $0 < \varepsilon \ll 1$, defined as the ratio of the characteristic lengths of both scales.

There are several examples of both natural and manufactured micro-heterogeneous materials, such as bones, soils, wood, paper, ceramics, concrete (Torquato, 2002) present in nature or developed for specific applications. Hence, studying such materials is clearly important, specially, how the interaction between their physical and geometrical properties occurs.

Separation of scales, together with the assumption of matter continuity at the local scale, guarantees the theoretical existence of an ideal homogeneous material equivalent to the micro-heterogeneous material in the sense that the constant properties of the former are the effective properties of latter (Bakhvalov & Panasenko, 1989). Observe that, as the properties of the equivalent homogeneous material are constant, its physical behavior is modeled by means of differential equations with constant coefficients. This approach is called *homogenization* and allows studying micro-heterogeneous materials via their equivalent homogeneous counterparts.

A particularly important case of micro-heterogeneous materials is that of microperiodic materials, whose structure is characterized by the periodic reproduction of a recurring element called a basic cell of periodicity of relative size ε. One way to approach mathematically this type of problem is through the use of the so-called asymptotic homogenization method (AHM) (Bakhvalov & Panasenko, 1989; Bensoussan et al., 1978; Ciouranescu & Donato, 2000; Tartar, 2009). The fundamental goal of the AHM is to obtain a formal asymptotic solution (FAS) (Bakhvalov & Panasenko, 1989) of the original problem with rapidly oscillating coefficients, namely, a two-scale series in terms of powers of ε with periodic coefficients in the local scale. The so-called homogenized problem for the first term of the FAS, which is independent of the local scale, models the behavior of the equivalent homogeneous material. The other coefficients are obtained in terms of the local functions, which are periodic in the local scale and are the solutions of the so-called local problems.

Recently, in the context of beams, the AHM approach proved to be a powerful tool for analyzing the steady-state behavior of microperiodic composite Euler-Bernoulli beams Huang et al. (2020), i.e. beams with piecewise constant properties. In this present work, we address the application of AHM to the more challenging case of twice continuously differentiable properties, which corresponds to functionally graded microperiodic Euler-Bernoulli beams. Here, a natural generalization of Silva et al. (2023) to the case of microperiodic load.

Finally, note that typical applications of the AHM deal with second-order differential equations. However, Euler-Bernoulli beams are modeled by means of fourth-order differential equations, so the usual tool (a Lemma of Bakhvalov & Panasenko (1989)) to prove the existence of the periodic solutions of the local problems, which allows constructing the FAS, are not applicable. So, we develop a fourth-order version of such a tool which, to the best of our knowledge, is original. This is done in section 2.1. Then, the original problem is formulated and solved in section 2.2, whereas the AHM is applied in section 2.3.

2 METHODOLOGY

2.1 Preliminaries

Lemma: Let $F(y)$ e $a(y)$ be 1-periodic differentiable functions, with $a(y)$ strictly positive and bounded. A necessary and sufficient condition for a 1-periodic solution $N(y)$ of the equation $\mathcal{L}N = F$, with $\mathcal{L} \equiv \dfrac{d^2}{dy^2}\left(a(y)\dfrac{d^2}{dy^2}\right)$, to exist is that $\langle F\rangle \equiv \displaystyle\int_0^1 F(y)dy = 0$.

In addition, the solution $N(y)$ is unique up to an additive constant, that is, $N(y,C) = \tilde{N}(y) + C$, where $\langle \tilde{N} \rangle = 0$ and C is a constant.

To the best of our knowledge, this result is original and its proof is described as follows. In order to prove necessity, it suffices to apply the averaging operator $\langle \cdot \rangle$ to the equation $\mathcal{L}N = F$. Then, it follows from the 1-periodicity of $a(y)$ and $N(y)$ that $\langle F \rangle = 0$. On the other hand, for proving sufficiency, an analytical expression for $N(y)$ is found, for which the values of the integration constants that ensure the 1-periodicity of $N(y)$ are only possible when $\langle F \rangle = 0$.

2.2 Problem formulation

Let ε be a parameter such that $0 < \varepsilon \ll 1$. Consider the problem of the mechanical equilibrium of a functionally graded Euler-Bernoulli beam of unit length, with microperiodic internal structure and flexural rigidity $a^\varepsilon \in \mathcal{C}^2(0,1)$, clamped at both ends and subjected to a distributed load $f^\varepsilon \in \mathcal{C}(0,1)$. This problem is stated as follows: for each ε, find the deflection $u^\varepsilon \in \mathcal{C}^4(0,1) \cap \mathcal{C}^1[0,1]$, solution of the differential equation

$$\mathcal{L}^\varepsilon u^\varepsilon \equiv \frac{d^2}{dx^2}\left(a^\varepsilon(x)\frac{d^2u^\varepsilon}{dx^2}\right) = f^\varepsilon(x), \quad x \in (0,1), \tag{1}$$

where $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$ is ε-periodic, strictly positive and bounded, and $f^\varepsilon(x) = f\left(x, \frac{x}{\varepsilon}\right)$ is ε-periodic in the second argument, subject to boundary conditions

$$u^\varepsilon(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=0} = 0, \quad u^\varepsilon(1) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 0. \tag{2}$$

The exact solution to the problem in (1)-(2) is obtained by direct integration as

$$u^\varepsilon(x) = \int_0^x \left(C_1^\varepsilon I_2^\varepsilon(w) + C_2^\varepsilon I_1^\varepsilon(w) + I_3^\varepsilon(w)\right) dw, \tag{3}$$

where

$$C_1^\varepsilon = \frac{1}{\Delta^\varepsilon}\int_0^1 \left(I_1^\varepsilon(1)I_3^\varepsilon(w) - I_3^\varepsilon(1)I_1^\varepsilon(w)\right) dw, \quad C_2^\varepsilon = \frac{1}{\Delta^\varepsilon}\int_0^1 \left(I_3^\varepsilon(1)I_2^\varepsilon(w) - I_2^\varepsilon(1)I_3^\varepsilon(w)\right) dw, \tag{4}$$

$$\Delta^\varepsilon = \int_0^1 \left(I_2^\varepsilon(1)I_1^\varepsilon(w) - I_1^\varepsilon(1)I_2^\varepsilon(w)\right) dw, \tag{5}$$

$$I_1^\varepsilon(w) = \int_0^w \frac{dr}{a^\varepsilon(r)}, \quad I_2^\varepsilon(w) = \int_0^w \frac{rdr}{a^\varepsilon(r)}, \quad I_3^\varepsilon(w) = \int_0^w \frac{1}{a^\varepsilon(r)}\int_0^r \int_0^s f^\varepsilon(t)dtdsdr. \tag{6}$$

Even though we have an analytical expression for the exact solution $u^\varepsilon(x)$ in (3)-(6), its actual calculation for particular cases can become analytically overly complex and otherwise computationally demanding. Therefore, we will opt for an alternative approach to find a good approximation of the exact solution, which has a simpler expression.

2.3 AHM application

Let $u^{(4)}(x,\varepsilon)$ be a formal asymptotic solution (FAS) of problem (1)-(2), which is an asymptotic expansion of the exact solution $u^\varepsilon(x)$ of (1)-(2) define as follows:

$$u^\varepsilon(x) \sim u^{(4)}(x,\varepsilon) = \sum_{k=0}^{4} \varepsilon^k u_k(x,y), \quad y = \frac{x}{\varepsilon}, \varepsilon = n^{-1}, n \in \mathbb{N}, \tag{7}$$

where $(x,y) \in (0,1)\times(0,n)$, and the coefficients of the powers of ε are unknown functions $u_k(x,y)$, $k \in \{0,1,2,3,4\}$, are continuously differentiable up to the fourth order with respect to both variables x and y, and 1-periodic in the local variable y.

By applying the second-order chain rule

$$\frac{d^2(\cdot)}{dx^2} = \frac{\partial^2(\cdot)}{\partial x^2} + 2\varepsilon^{-1}\frac{\partial^2(\cdot)}{\partial x\partial y} + \varepsilon^{-2}\frac{\partial^2(\cdot)}{\partial y^2}. \tag{8}$$

to the differential equation (1) of the original problem, we get

$$\left(\mathcal{L}^{xx}_{xx} + \varepsilon^{-1}(2\mathcal{L}^{xx}_{xy} + 2\mathcal{L}^{xy}_{xx}) + \varepsilon^{-2}(\mathcal{L}^{xx}_{yy} + 4\mathcal{L}^{xy}_{xy} + \mathcal{L}^{yy}_{xx}) + \varepsilon^{-3}(2\mathcal{L}^{xy}_{yy} + 2\mathcal{L}^{yy}_{xy}) + \varepsilon^{-4}\mathcal{L}^{yy}_{yy}\right) u^\varepsilon = f, \tag{9}$$

where the linear differential operators $\mathcal{L}^{\alpha\beta}_{\gamma\varphi}$, are defined as

$$\mathcal{L}^{\alpha\beta}_{\gamma\varphi}(\cdot) = \frac{\partial^2}{\partial\alpha\partial\beta}\left(a(y)\frac{\partial^2(\cdot)}{\partial\gamma\partial\varphi}\right), \quad \alpha,\beta,\gamma,\varphi \in \{x,y\}, \tag{10}$$

with the 1-periodic flexural rigidity $a(y)$ with respect to the local variable y.

By substituting (7) into (9), and rearranging the terms by powers of ε, we obtain

$$
\begin{aligned}
&\varepsilon^{-4}\mathcal{L}_{yy}^{yy}u_0\\
&+\varepsilon^{-3}\left(\mathcal{L}_{yy}^{yy}u_1+2\mathcal{L}_{yy}^{xy}u_0+2\mathcal{L}_{xy}^{yy}u_0\right)\\
&+\varepsilon^{-2}\left(\mathcal{L}_{yy}^{yy}u_2+2\mathcal{L}_{yy}^{xy}u_1+2\mathcal{L}_{xy}^{yy}u_1+\mathcal{L}_{yy}^{xx}u_0+4\mathcal{L}_{xy}^{xy}u_0+\mathcal{L}_{xx}^{yy}u_0\right)\\
&+\varepsilon^{-1}\left(\mathcal{L}_{yy}^{yy}u_3+2\mathcal{L}_{yy}^{xy}u_2+2\mathcal{L}_{xy}^{yy}u_2+\mathcal{L}_{yy}^{xx}u_1+4\mathcal{L}_{xy}^{xy}u_1+\mathcal{L}_{xx}^{yy}u_1+2\mathcal{L}_{xy}^{xx}u_0+2\mathcal{L}_{xx}^{xy}u_0\right)\\
&+\varepsilon^{0}\left(\mathcal{L}_{yy}^{yy}u_4+2\mathcal{L}_{yy}^{xy}u_3+2\mathcal{L}_{xy}^{yy}u_3+\mathcal{L}_{yy}^{xx}u_2+4\mathcal{L}_{xy}^{xy}u_2+\mathcal{L}_{xx}^{yy}u_2+2\mathcal{L}_{xy}^{xx}u_1+2\mathcal{L}_{xx}^{xy}u_1+\mathcal{L}_{xx}^{xx}u_0-f\right)\\
&=\mathcal{O}(\varepsilon)
\end{aligned}
\tag{11}
$$

So, in order for the asymptotic equality in (11) to hold true, the existence of solutions u_k, $k\in\{0,1,2,3,4\}$, 1-periodic in y, to the following recurrence of differential equations must be guaranteed:

$$
\begin{aligned}
\varepsilon^{-4}&:\mathcal{L}_{yy}^{yy}u_0=0,\\
\varepsilon^{-3}&:\mathcal{L}_{yy}^{yy}u_1=-2\mathcal{L}_{yy}^{xy}u_0-2\mathcal{L}_{xy}^{yy}u_0,\\
\varepsilon^{-2}&:\mathcal{L}_{yy}^{yy}u_2=-2\mathcal{L}_{yy}^{xy}u_1-2\mathcal{L}_{xy}^{yy}u_1-\mathcal{L}_{yy}^{xx}u_0-4\mathcal{L}_{xy}^{xy}u_0-\mathcal{L}_{xx}^{yy}u_0,\\
\varepsilon^{-1}&:\mathcal{L}_{yy}^{yy}u_3=-2\mathcal{L}_{yy}^{xy}u_2-2\mathcal{L}_{xy}^{yy}u_2-\mathcal{L}_{yy}^{xx}u_1-4\mathcal{L}_{xy}^{xy}u_1-\mathcal{L}_{xx}^{yy}u_1-2\mathcal{L}_{xy}^{xx}u_0-2\mathcal{L}_{xx}^{xy}u_0,\\
\varepsilon^{0}&:\mathcal{L}_{yy}^{yy}u_4=-2\mathcal{L}_{yy}^{xy}u_3-2\mathcal{L}_{xy}^{yy}u_3-\mathcal{L}_{yy}^{xx}u_2-4\mathcal{L}_{xy}^{xy}u_2-\mathcal{L}_{xx}^{yy}u_2-2\mathcal{L}_{xy}^{xx}u_1-2\mathcal{L}_{xx}^{xy}u_1-\mathcal{L}_{xx}^{xx}u_0+f.
\end{aligned}
\tag{12}
$$

Note that, for each fixed $x\in(0,1)$, the equations in the recurrence (12) are of the form $\mathcal{L}N=F$ with $F(y)$ and $a(y)$ being 1-periodic, where the 1-periodicity of $a(y)$ is inherited from the ε-periodicity of $a^{\varepsilon}(x)$. Thus, noting that for fixed x, $\mathcal{L}_{yy}^{yy}\equiv\mathcal{L}$, the Lemma can be applied to guarantee the existence (and also uniqueness) of 1-periodic solutions of the recurrence of equations in (12).

Applying the Lemma to the problem for $u_0(x,y)$ in (12) with $N\equiv u_0$ and $F\equiv 0$, we have that the existence of a 1-periodic solution u_0 in y is guaranteed, and also that u_0 does not depend on y, i.e.,

$$
u_0(x,y)=u_0(x),
\tag{13}
$$

so $u_0(x)$ represents the mean deflection, which is independent of the microstructure, i.e., independent of the local variable y and the geometrical parameter ε. Then, considering

(13), we update the problem for $u_1(x,y)$ in (12) and obtain

$$\mathcal{L}_{yy}^{yy} u_1 = 0, \tag{14}$$

which is identical to in (12). Thus, from (14), we have $N \equiv u_1$ and $F \equiv 0$, so by the Lemma, the existence of the solution u_1, which is 1-periodic in y, is guaranteed, and it follows that u_1 also does not depend on y, i.e., $u_1(x,y) = u_1(x)$. However, $u_1(x)$ also represents a contribution to the mean deflection, which is perturbed by ε in FAS (7). So, in order to comply with the mean deflection being independent of the microstructure, the only admissible realization of $u_1(x)$ is

$$u_1(x) = 0, \tag{15}$$

that is, applying the mean value operator $\langle \cdot \rangle$ to (7) yields the following estimation of the mean deflection:

$$\langle u^\varepsilon(x) \rangle \sim \left\langle u^{(4)}(x,\varepsilon) \right\rangle = u_0(x) + \varepsilon u_1(x) + \sum_{k=2}^{4} \varepsilon^k \left\langle u_k(x,y) \right\rangle, \tag{16}$$

which implies (15) and conditions

$$\langle u_k(x,y) \rangle = 0, \quad k \in \{2,3,4\}. \tag{17}$$

Updating the problem for $u_2(x,y)$ in (12), based on (13) and (15), we have

$$\mathcal{L}_{yy}^{yy} u_2 = -\frac{d^2 a}{dy^2} \frac{d^2 u_0}{dx^2}, \tag{18}$$

from which, identifying $N \equiv u_2$ and $F \equiv -\frac{d^2 a}{dy^2}\frac{d^2 u_0}{dx^2}$, the Lemma ensures the existence of $u_2(x,y)$, 1-periodic in y, since $\langle F \rangle = 0$, due to the 1-periodicity in y of $a(y)$. Thus, considering the structure of the right-hand side of (18), we take

$$u_2(x,y) = N_2(y) \frac{d^2 u_0}{dx^2}, \tag{19}$$

where local function $N_2(y)$ is 1-periodic. Substituting (19) into (18) and (17) with $k = 2$ assuming $\frac{d^2 u_0}{dx^2} \neq 0$, we obtain that $N_2(y)$ is the 1-periodic solution of the so-called *first* *local problem* defined by the differential equation

$$\frac{d^2}{dy^2}\left(a(y)\frac{d^2N_2}{dy^2}+a(y)\right)=0, \tag{20}$$

and the condition $\langle N_2(y)\rangle = 0$, which leads to

$$N_2(y)=\int_0^y\int_0^t\left(\frac{\widehat{a}}{a(s)}-1\right)dsdt+C_1y+C_2, \tag{21}$$

where $\widehat{a}=\langle (a(y))^{-1}\rangle^{-1}$ is the so-called *effective coefficient of flexural rigidity*, and

$$C_1=-\left\langle \int_0^y\left(\frac{\widehat{a}}{a(s)}-1\right)ds\right\rangle, \quad C_2=-\left\langle \int_0^y\int_0^t\left(\frac{\widehat{a}}{a(s)}-1\right)dsdt+C_1y\right\rangle.$$

It follows from (13), (15), and (19) that the problem for $u_3(x,y)$ can be updated as

$$\mathcal{L}_{yy}^{yy}u_3=-\frac{d^2}{dy^2}\left(a(y)\frac{dN_2}{dy}\right)\frac{d^3u_0}{dx^3}, \tag{22}$$

from which, identifying $N\equiv u_3$ and $F\equiv -\frac{d^2}{dy^2}\left(a(y)\frac{dN_2}{dy}\right)\frac{d^3u_0}{dx^3}$, it follows that $\langle F\rangle = 0$, as $a(y)$ and $N_2(y)$ are 1-periodic. Therefore, the Lemma guarantees the existence of the solution $u_3(x,y)$, which is 1-periodic in y. Thus, considering the structure of the right-hand side of (22), we take

$$u_3(x,y)=N_3(y)\frac{d^3u_0}{dx^3}, \tag{23}$$

where local function $N_3(y)$ is 1-periodic. Substituting (23) into (22) and (17) with $k=3$, assuming $\frac{d^3u_0}{dx^3}\neq 0$, we obtain that $N_3(y)$ is a 1-periodic solution of the so-called *second local problem* defined by the differential equation

$$\frac{d^2}{dy^2}\left(a(y)\frac{d^2N_3}{dy^2}+2a(y)\frac{dN_2}{dy}\right)=0, \tag{24}$$

and the condition $\langle N_3(y)\rangle = 0$, from which, using $\langle N_2(y)\rangle = 0$ again, we have

$$N_3(y)=-2\int_0^y N_2(s)ds+C_3, \tag{25}$$

where

$$C_3 = 2\left\langle \int_0^y N_2(s)ds \right\rangle.$$

Finally, applying the Lemma to the equation for $u_4(x,y)$ in the recurrence (12), using (13), (15), (19) and (23), we obtain that the condition for the existence of a solution $u_4(x,y)$, which is 1-periodic in y, is that $u_0(x)$ is the solution of the so-called *homogenized problem* defined by the *homogenized equation*

$$\mathcal{L}^0 u_0 \equiv \widehat{a}\frac{d^4u_0}{dx^4} = \widehat{f}(x), \tag{26}$$

where $\widehat{f}(x) = \langle f(x,y)\rangle$ is the mean load, and the boundary conditions by substituting FAS (7) into the original boundary conditions (2):

$$u_0(0) = 0, \quad \left.\frac{du_0}{dx}\right|_{x=0} = 0, \quad u_0(1) = 0, \quad \left.\frac{du_0}{dx}\right|_{x=1} = 0, \tag{27}$$

so, the solution $u_0(x)$ of problem (26)-(27) is obtained by direct integration as

$$u_0(x) = \frac{1}{\widehat{a}}\left(\widehat{C}_1x^3 + \widehat{C}_2x^2 + \int_0^x \widehat{I}(w)dw\right), \tag{28}$$

where

$$\widehat{C}_1 = 2\int_0^1 \widehat{I}(w)dw - \widehat{I}(1), \quad \widehat{C}_2 = \widehat{I}(1) - 3\int_0^1 \widehat{I}(w)dw, \quad \widehat{I}(w) = \int_0^w\int_0^r\int_0^s \widehat{f}(t)dtdsdr. \tag{29}$$

Therefore, being u_0 the solution of the equation of the homogenized problem, so (26) becomes an identity, and taking account (20) and (25), we have that the problem for $u_4(x,y)$ can be rewritten as

$$\mathcal{L}_{yy}^{yy}u_4 = -3\frac{d^2}{dy^2}\left(a(y)N_2(y)\right)\frac{d^4u_0}{dx^4} + \widehat{f}(x) - f(x,y). \tag{30}$$

Then, considering the structure of the right-hand side of (30), we take

$$u_4(x,y) = N_{41}(y)\frac{d^4u_0}{dx^4} + N_{42}(x,y), \tag{31}$$

where local functions $N_{41}(y)$ and $N_{42}(x,y)$ are 1-periodic in y. Substituting (31) into (30) assuming $\dfrac{d^4u_0}{dx^4} \neq 0$, we obtain that $N_{41}(y)$ and $N_{42}(x,y)$ are the 1-periodic solutions of the so-called *third local problems* defined by the differential equations

$$\frac{d^2}{dy^2}\left(a(y)\frac{d^2N_{41}}{dy^2} + 3a(y)N_2(y)\right) = 0, \qquad \frac{\partial^2}{\partial y^2}\left(a(y)\frac{\partial^2 N_{42}}{\partial y^2}\right) = \widehat{f}(x) - f(x,y), \tag{32}$$

and conditions $\langle N_{41}(y)\rangle = 0$ and $\langle N_{42}(x,y)\rangle = 0$, respectively. Then, we have

$$N_{41}(y) = 3\int_0^y\int_0^s N_2(t)dtds + C_4y + C_5, \tag{33}$$

$$N_{42}(x,y) = \int_0^y\int_0^z \frac{1}{a(w)}\left(\int_0^w\int_0^s \left(\widehat{f}(x) - f(x,t)\right)dt\,ds + C_6w + C_7\right)dwdz + C_8y + C_9, \tag{34}$$

where

$$\begin{aligned}
&C_4 = -3\left\langle \int_0^y N_2(t)dt\right\rangle, \quad C_5 = -\left\langle 3\int_0^y\int_0^s N_2(t)dtds + C_4y\right\rangle,\\
&C_6 = -\left\langle \int_0^y \left(\widehat{f}(x) - f(x,t)\right)dt\right\rangle, \quad C_7 = -\left\langle \frac{1}{a(y)}\int_0^y\int_0^s \left(\widehat{f}(x) - f(x,t)\right)dtds\right\rangle - C_6\frac{\widehat{a}}{2},\\
&C_8 = -\left\langle \int_0^y \frac{1}{a(w)}\left(\int_0^w\int_0^s \left(\widehat{f}(x) - f(x,t)\right)dt\,ds + C_6w + C_7\right)dw\right\rangle,\\
&C_9 = -\left\langle \int_0^y\int_0^z \frac{1}{a(w)}\left(\int_0^w\int_0^s \left(\widehat{f}(x) - f(x,t)\right)dt\,ds + C_6w + C_7\right)dwdz\right\rangle - \frac{C_8}{2}.
\end{aligned} \tag{35}$$

Therefore, substituting (13), (15), (19), (23), and (31) into (7), we obtain the following expression for the FAS, returning to the original variable:

$$u^{(4)}(x,\varepsilon) = u_0(x) + \varepsilon^2 N_2\left(\frac{x}{\varepsilon}\right)\frac{d^2u_0}{dx^2} + \varepsilon^3 N_3\left(\frac{x}{\varepsilon}\right)\frac{d^3u_0}{dx^3} + \varepsilon^4\left(N_{41}\left(\frac{x}{\varepsilon}\right)\frac{d^4u_0}{dx^4} + N_{42}\left(x,\frac{x}{\varepsilon}\right)\right). \tag{36}$$

Observe that FAS (36) does not satisfies the original boundary conditions (2). In fact, among the lower-order FASs $u^{(k)}(x,\varepsilon)$, $k \in \{0,2,3\}$, obtained by truncating (7), zeroth-order FAS $u^{(k)}(x,\varepsilon) \equiv u_0(x)$ is the only that satisfies conditions (2).

3 CONCLUSIONS

Based on the above, it is evident that the use of the AHM is highly beneficial for dealing with problems involving coefficients that oscillate rapidly with respect to position, as it is not always possible to obtain the exact solution to such problems using traditional methods. Although the original problem addressed here has an expression for the exact solution, its implementation becomes nontrivial due to the nature of the coefficient $a^{\varepsilon}(x)$. Thus, one of the advantages of using AHM is that the problem to obtain the solution $u_0(x)$ is, at least in structure, simpler than the problem for the exact solution $u^{\varepsilon}(x)$, as evidenced by the respective differential equations (1) and (26). This can also be observed by considering that, in the original problem, the beam is considered non-uniform, whereas in the homogenized problem, uniformity is guaranteed. Additionally, the AHM approach allows for a deeper understanding of the beam's behavior in relation to spatial variations of the coefficients, thus facilitating the analysis and resolution of practical structural engineering problems.

It is worth noting that this is an initial work in the field, but it encompasses all the fundamental steps of applying the AHM, standing out for the originality of the Lemma presented and the richness of details provided.

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REFERENCES

Bakhvalov, N. & Panasenko, G. (1989). *Homogenisation: Averaging Processes in Periodic Media: Mathematical Problems in the Mechanics of Composite Material*. (MASS Series, 1st ed.). Springer Netherlands.

Bensoussan, A., Lions, J., & Papanicolau, G. (1978). *Asymptotic Analysis for Periodic Structures*. (MAEE series, 5th vol.). North Holland.

Ciouranescu, D. & Donato, P. (2000). *An Introduction to Homogenization*. (17th vol.). Oxford University Press.

Huang, Z., Xing, Y., & Gao, Y. (2020). A two-scale asymptotic expansion method for periodic composite Euler beams. *Composite Structures*, *241*:112033.

Rao, S. S. (2016). *Mechanical Vibrations*. (6th ed.). Harlow: Pearson.

Silva, D. M., Pérez-Fernández, L. D., Molter, A., & Bravo-Castillero, J. (2023). Método de homogeneização assintótica aplicado ao modelo de vigas de Euler-Bernoulli estático. *Anais da Jornada de Matemática e Matemática Aplicada*. São Paulo, SP, Brazil, 4.

Tartar, L. (2009). *The General Theory of Homogenization: A Personalized Introduction*. UMILN. (UMILN series, 7th vol.). Springer.

Torquato, S. (2002). *Random Heterogeneus Materials: Microstructure and Macroscopic properties*. (IAM series, 16th ed.). Springer.

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