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On a level-set regularization strategy for the identification of a piecewise constant rigidity coefficient in a beam

Uma estratégia de regularização tipo level-set para a identificação de um coeficiente de rigidez constante por partes em uma viga

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ABSTRACT

In this work, we analyze a level set regularization strategy to identify the constant piecewise stiffness coefficient in a static Euler-Bernoulli beam, based on indirect deflection measurements. The theoretical results presented are illustrated by some numerical simulations.

Keywords: Level-set regularization; Piecewise constant stiffness coefficient; Beams; Euler-Bernoulli

RESUMO

Neste trabalho, analisamos uma estratégia de regularização tipo level-sets para a identificação do coeficiente de rigidez constante por partes em uma viga do tipo Euler-Bernoulli estática, a partir de medidas indiretas da deflexão. Os resultados teóricos apresentados são ilustrados por algumas simulações numéricas.

Palavras-chave: Regularização level-sets; Coeficiente de rigidez descontínuo; Vigas; Euler-Bernoulli

1 INTRODUCTION

Beams are a key component of structural systems and are essential for supporting engineering projects (Hibbeler, 2006). In many practical situations, beams are constantly impacted by environmental loads, caused by environmental phenomena such as wind, waves, current, tides, earthquakes, temperature, ice, seabed movement, and marine growth, among others (Hibbeler, 2006). These phenomena usually change the main structural components of the beam, leading to future collapse. For the sake of structural safety, it is important to determine whether there are damages or changes to the materials that make up these structures. It is often too costly, if not impossible, to measure alterations in the characteristics of beam materials directly. On the contrary, alterations in the physical characteristics of materials can frequently lead to modifications in the stiffness coefficient $a(x) = E(x)I(x) > 0$, resulting from the product of the modulus of elasticity $E(x)$ and the moment of inertia $I(x)$. As a result, the beam resistance to deflection is affected under bending moments (Hibbeler, 2006).

Therefore, a key problem associated with beam theory in engineering is the identification of the stiffness coefficients $a(x)$ of the beams in a non-destructive manner. This is termed the inverse problem in beam theory (Kawano, 2017; Lesnic et al., 1999; Medeiros et al., 2022). This technique allows the evaluation of the risks and measures to be adopted in the structure, based on indirect measurements of the beam of the transverse deflection.

More precisely, in this work, we assume that the beam is one-dimensional and of length L. The beam is loaded by a known transverse force $f(x) \in L^2[0, L]$ which results in a deflection $u(x)$ modeled by the Euler-Bernoulli type equation

$$M''(x) = f(x)\,, \qquad a(x)u''(x) = M(x) \tag{1}$$

with boundary conditions

$$u(0) = u(L) = M(0) = M(L) = 0\,, \tag{2}$$

characterizing the beam in question as being statically determined (Hibbeler, 2006).

The determination of the deflection $u = u(x)$ that satisfies the equation (1), with boundary conditions (2), knowing the stiffness coefficient $a(x)$ and the loading force $f(x)$ is a traditional mechanical problem, also known as a direct problem (Hibbeler, 2006).

The focus of this contribution is on the inverse problem related to the Euler-Bernoulli beam. In fact, identification of the stiffness coefficient $a(x)$ of the beam, which $a(x)$ is piecewise constant, given that the loading force $f \in L^2[0, L]$ and the boundary conditions (2), from indirect (non-destructive) measurements of the deflection of $u(x)$, for $x \in [0, L]$. In other words, the available measurements are given by u^δ, subject to level errors $\delta \geq 0$, of the deflection $u(x)$, which satisfying

$$\|u - u^\delta\|^2_{L^2[0,L]} \leq \delta \,. \tag{3}$$

A characteristic shared by almost all the inverse problems, which includes the identification of the stiffness coefficient $a(x)$ (see Theorem 3), is that they are ill-posed in the sense of Hadamard (Kirsh, 2011). Therefore, the stability of the proposed identification with respect to noise measurements is one of the main concerns (Kirsh, 2011).

In this contribution, we will propose and analyze a regularization method, called leve-set, for the stable identification of the stiffness coefficient $a(x)$, when it assumes two positive, known, and distinct values, that is, $a(x) \in \{a_1, a_2\}$ almost always in $[0, L]$. In other words, we assume that there exist sets of non-zero measure $\Omega_1, \Omega_2 \subset [0, L]$, with finite Hausdorff measure and $[0, L]\Omega_1 \cup \Omega_2$, such that $a(x) = a_1$ if $x \in \Omega_1$ and $a(x) = a_2$ if $x \in \Omega_2 := [0, L] - \Omega_1$.

It is worth mentioning that with such assumptions, the stiffness coefficient $a(x)$ can be rewritten as

$$a(x) = a_2 + (a_1 - a_2)\chi_{\Omega_1} \,, \tag{4}$$

where χ_X represents the characteristic function of the set X.

Literature overview: Extensive research has recently been conducted on the identification of stiffness coefficients in the Euler-Bernoulli beam theory; see (Gladwell & Morassi, 2010; Kawano, 2017; Lerma & Hinestroza, 2017; Lesnic et al., 1999; Marinov et al., 2015; Marinov & Vatsala, 2008; Medeiros, 2019; Medeiros et al., 2022) and references. In this section, we provide a summary of the papers that discuss methods for regularizing the estimation of $a(x)$ based on indirect observations of the beam deflection $u(x)$ within the framework of Euler-Bernoulli beam theory.

In Lesnic et al. (1999), the authors demonstrate uniqueness identification and a Tikhonov-type regularization approach for the reconstruction of a smooth ($C^1[0, L]$) rigidity coefficient $a(x)$, from measurements of beam deflection. However, the smoothness of the coefficient required for the analysis conducted in Lesnic et al. (1999) implies that intriguing physical scenarios, such as crack beams, where $a(x)$ is represented as in (4), cannot be taken into account.

In Marinov et al. (2015), a formula was provided to obtain a piecewise constant coefficient $a(x)$. Marinov & Vatsala (2008) then assumed the stiffness to be a piecewise polynomial function and used a similar technique to Marinov et al. (2015) to identify the polynomial coefficients. However, this technique requires prior knowledge of where the coefficients change their values, which is not practical as it implies knowing where the beam has failed beforehand. Additionally, the equation obtained depends on a quotient that involves the second derivative of the data, which is an ill-posed problem (Kirsh, 2011).

In Lerma & Hinestroza (2017), a method of reconstructing the stiffness coefficient $a(x)$ is presented that involves the use of Green's functions. This approach yields a smooth $a(x)$ that is reconstructed.

In Kawano (2017), a unique solution is presented to the issue of determining the flexural stiffness in a dynamic Euler-Bernoulli beam, based on the observation of boundary measurements. The author of Kawano (2017) shows how, by observing the displacement and slope of a vibrating beam at one of its ends for a brief period, the unique flexural stiffness $a(x)$ can be identified.

Investigation of parameter identification in the Euler-Bernoulli beam theory is also conducted as an inverse spectral problem, as seen in (Gladwell & Morassi, 2010) and other related sources. These approaches are faced with the challenge of obtaining spectral data as measured output.

In Medeiros (2019); Medeiros et al. (2023,2), the uniqueness identification of a non-necessarily smooth stiffness coefficient $a(x)$ as in (4) is proved. Then, iterative regularization approaches are used in the reconstructions. It turns out that the smoothness of the beam solution implies that the iterative reconstructed $a(x)$ is smooth as well (see Section 3 for details).

Article novelties and organization: In this work, it is shown that it is possible to use techniques such as level sets (e.g. (Cezaro et al., 2009)) as a regularization method to identify the piecewise constant stiffness coefficient $a(x)$ as in (4), in the Euler-Bernoulli equation (1). Some numerical simulations obtained from the iterative method generated by level sets techniques are presented to support the theoretical results. Such results appear linked in the text as follows: In Section 2, the problem of identifying the coefficient $a(x)$ is formulated as a parameter-to-measures operator. It is proved that this operator is continuous and Fréchet differentiable with respect to the topology of $L^2[0, L] \cap L^1[0, L]$. On the one hand, these results show that any regularization method based on adjoint equations such as classical Tikhonov (see (Lesnic et al., 1999)) or iterative regularization (see (Medeiros, 2019; Medeiros et al., 2023,2)) results in a smooth reconstructed $a(x)$. We demonstrate that the level set formulation presented in Section 4 is a regularization strategy for the problem. In Section 4.2, the algorithm derived from the proposed level set technique is presented, which will be exemplified with some numerical simulations. Section 6 presents some conclusions.

Main assumptions: In the forthcoming analysis we shall assume that:

(H1) The flexural stiffness coefficient $a(x) \in Ad := \{a \text{ measurable on} [0, L], 0 < \underline{a} \leq a(x) \leq \overline{a}\}$, for known constants $\overline{a}, \underline{a}$. In particular, the piecewise constant $a(x)$ as in (4) satisfies **H1)**.

(H2) The load distribution $f \in L^2[0, L]$.

(H3) The measurements $u^\delta \in L^2[0, L]$ differ from exact data $u \in L^2[0, L]$ from a noisy level $\delta > 0$, that is, $\|u - u^\delta\|_{L^2[0,L]} \leq \delta$.

(H4) The problem (6) has a solution $a^* \in Ad$.

2 PARAMETER-TO-MEASUREMENT MAP AND ITS PROPERTIES

In this section, we introduce the flexural stiffness coefficient identification problem as a non-linear operator equation, called the parameter-to-solution map. Many results in this section were already presented in (Medeiros, 2019; Medeiros et al., 2023,2), but they are important to understand why Tikhonov's or iterative regularization proposed in (Lesnic et al., 1999; Medeiros et al., 2023,2) produces a smooth approximate stiffness coefficient $a(x)$. Therefore, the level-set regularization approach proposed in the following sections is an alternative to obtain a regularized solution of a piecewise constant $a(x)$ as in (4).

We begin by introducing the weak solution of the system (1) with boundary conditions (2).

Lemma 1. *Let assumptions **(H1)** and **(H2)** be satisfied. Then, there exists a unique weak solution* $u \in H^2[0, L] \cap H_0^1[0, L]$ *of* (1)*, with boundary conditions* (2)*.*

Here, a weak solution is any $u \in H^1[0, L]$ *that satisfies*

$$\int_0^L u'\phi' dx = \int_0^L \frac{M}{a}\phi dx \qquad \forall \phi \in H_0^1[0, L]\,. \tag{5}$$

Proof. It is deduced from Assumption **(H2)**, the boundary conditions (2) and the elliptic regularity (Brezis, 2010) that a unique $M \in C[0, L] \cap H^2[0, L] \subset L^2[0, L]$ is the solution of the first equation in (1). The continuity of M implies that it is bounded uniformly in the interval $[0, L]$. Furthermore, due to assumption **H1)** the bi-linear form related to the variational formulation (5) and the corresponding linear functional in the right-hand-side of (5) are both continuous and coercive in $H^1[0, L]$. Therefore, it the existence and uniqueness of a weak solution of (1)-(2) is deduced from Lax-Milgram's lemma and the elliptic regularity theory, as stated in (Brezis, 2010). □

As a consequence of Lemma 1, the parameter-to-solution map (forward operator), defined by,

$$F : Ad \subset L^2[0, L] \longrightarrow L^2[0, L] \tag{6}$$

$$a \longmapsto F(a) = u(a)\,, \tag{7}$$

where, $u(a)$ is the unique weak solution of (1)-(2), is well defined.

We demonstrate certain characteristics of the operator (6) that are essential for comprehending the ill-posedness and suggest regularization techniques for the inverse problem of the stiffness coefficient.

The following theorem demonstrates the continuity of the operator (6) in the $L^2[0,L]$ space, which is our first result in this direction.

Theorem 2. *Let assumptions **(H1)** and **(H2)** be satisfied. Consider the boundary conditions* (2)*. Let the parameter-to-solution map F defined in* (6)*. Then:*

i) *F is continuous from $Ad \subset L^2[0,L]$ to $L^2[0,L]$.*

ii) *F is also continuous in the $L^1[0,L]$ norm topology.*

Proof. Observe that since $L^2[0,L]$ is a metric space, it is sufficient to show that F is sequentially continuous, e.g. (Brezis, 2010). Let $(a_n) \in Ad$ be a convergent sequence to a in $L^2[0,L]$. Since Ad is closed, it follows that $a \in Ad$. Denote by $F(a_n) = u(a_n) := u_n$ and $F(a) = u(a) := u$ the unique weak solutions of (1) (2) (see Lemma 1), corresponding to $a_n, a \in Ad$, respectively.

The linearity of (1), implies that $u_n - u$ satisfies

$$(u_n - u)'' = -\frac{(a_n - a)}{a} u_n'' = -\frac{(a_n - a)}{a_n a} M. \tag{8}$$

with homogeneous boundary condition.

From the assumptions **(H1)** and **(H2)**, it is deduced that $\|M\| < \tilde{C}$ and $\|a_n a\| > \underline{a}^2$. Therefore, $\frac{M}{a_n a}$ is uniformly bounded in $[0,L]$. Integrating (8) with $v_n := u_n - u \in H_0^1[0,L]$ as the test function, and using the Poincaré and Hölder inequality, we obtain

$$\|v_n\|_{L^2[0,L]} \leq C\|a_n - a\|_{L^2[0,L]} \tag{9}$$

where C is a constant that depends only on the Poincaré's constant, $\|M\|_{L^\infty[0,L]}$ and $\underline{a}^2$.

It follows from (9) assertion i) is satisfied.

Furthermore, from the assumption **H1)**, we have $|a(x)| \leq \overline{a}$ for x a.e. in $[0,L]$, $M > 0$. Then, $a \in L^s[0,L]$ for any $1 \leq s < \infty$ and

$$\|a\|^s_{L^s[0,L]} = \int_\Omega |a(x)|^{s-1}|a(x)|\, dx \leq \overline{a}^{s-1}\|a\|_{L^1[0,L]}\,. \tag{10}$$

In particular, equation (10) implies in

$$\|a\|_{L^s[0,L]} \leq \overline{a}^{(s-1)/s}\|a\|^{1/s}_{L^1[0,L]}. \tag{11}$$

As a consequence of item i), (9) and (11), the assertion on item ii) follows. □

We will demonstrate that the stiffness identification problem we are considering is ill-posed, as stated in (Kaltenbacher et al., 2008; Kirsh, 2011). This means that even slight changes in measurements u^δ can have a major impact on identifying the stiffness coefficient a. This is due to the compactness of the operator F defined in (6) in the $L^2[0,L]$-topology, because if the inverse operator F^{-1} exists, it is not limited. We will present this result in the following theorem.

Theorem 3. *Suppose that* ***(H1)*** *and* ***(H2)*** *are fulfilled. Additionally, assume that the boundary conditions in* (2) *are satisfied. Consequently, the operator F specified in* (6) *is compact from $Ad \subset L^2[0,L]$ to $L^2[0,L]$.*

Proof. Let $(a_n) \in Ad$ be the sequence that weakly converges to a in the $L^2[0,L]$ norm topology. The convexity and closedness of Ad, implies that it is weakly closed (Brezis, 2010). As a consequence, $a \in Ad$. Let the notation for $F(a_n) = u_n$ and $F(a) = u$ as in Theorem 2. Arguing as in Theorem 2, we have that the difference $F(a_n) - F(a) := u_n - u$ satisfies the variational formulation given by

$$\int_0^L a_n(u_n - u)''\varphi dx = -\int_0^L (a_n - a)aM\varphi dx, \tag{12}$$

for any test function $\varphi \in C_0^\infty[0,L]$.

Furthermore, assumptions **(H1)** and **(H2)** implies that $aM \in L^\infty[0,L]$. As a result, $aM\varphi \in L^2[0,L]$. Consequently, due to the weak convergence of a_n to a in $L^2[0,L]$, the right side of (12) tends to zero as n increases. As a result, the left side of (12) converges to zero in the sense of distributions.

Under the assumption **H1)**, the dominated convergence theorem (Brezis, 2010) and the fact that $C_0^\infty[0,L]$ is dense in $L^2[0,L]$ imply that $u_n - u \rightharpoonup z$ in $L^2[0,L]$. Furthermore, z satisfies

$$z'' = 0\,, \tag{13}$$

with homogeneous boundary conditions, in the sense of distributions. Then, the maximum principle (Brezis, 2010) applied to (13) and the fact that $a \in Ad$ implies $z \equiv 0$. Consequently, we have $u_n - u \xrightarrow{L^2} 0$.

On the other hand, substituting $\varphi_n = u_n - u$ into (12) and considering Assumption **(H1)**, we can deduce that $\|u_n - u\|_{L^2} \to 0$. This implies that $u_n - u$ converges weakly and in norms to zero, which in turn implies that $u_n \to u$ in $L^2[0, L]$, thus concluding the proof. □

Theorem 3 suggests that a regularization technique should be used to ensure a reliable determination of the stiffness coefficient $a(x)$ from measurable data (noise affected), for example, as in (Kaltenbacher et al., 2008; Kirsh, 2011).

In the following we show the results for with we are able to deduce that classical Tikhonov or iterative regularization as proposed in (Lesnic et al., 1999; Medeiros et al., 2023,2) implies smooth reconstructed stiffness coefficients $a(x)$. For that fate, define the following auxiliary operator

$$A(a) : H^2[0, L] \cap H_0^1[0, L] \longrightarrow L^2[0, L] \tag{14}$$

$$u \longmapsto A(a)u := -au''. \tag{15}$$

for any $u = u(a)$ the unique solution of (1) with the boundary conditions (2), for $a \in Ad$. It is noteworthy that Lemma 1 implies that $A(a)$ is properly defined.

The following proposition is about the Fréchet differentiability of the parameter-to-solution map $F(a)$ defined in (6). Furthermore, we define the corresponding problem that the Fréchet derivative $F'(a)$ and its adjoint $F'(a)^*$ must satisfy.

Proposition 4. *Consider the Assumptions **(H1)** and **(H2)** be satisfied. Then:*

i) The operator $A(a)$ defined in (14) *is linear, bounded, and invertible. Its inverse $A^{-1}(a)$ is also linear and bounded.*

ii) The operator $A(a)$ defined in (14) *has a linear and bounded adjoint operator defined by $A^*(a)v = -(av)''$. Furthermore, $A^*(a)$ has a linear bounded inverse given by $(A^*(a))^{-1} := (A^{-1}(a))^*$, such that $(A^*(a))^{-1}w = v$, where v is the unique solution of $-(av)'' = w$.*

iii) The operator $F(a)$ *in* (6) *is Fréchet differentiable. Its Fréchet derivative* $F'(a)$ *is a linear and bounded operator satisfying*

$$F'(a)h = A^{-1}(a)[hu''], \tag{16}$$

with

$$\|F'(a)\| \leq \underline{a}^{-1}\|M\|_{L^2[0,L]} := \gamma^{-1}\,. \tag{17}$$

iv) The adjoint of $F'(a)$ *is also a bounded linear operator that satisfies*

$$F'(a)^*r = u''(a)(A^{-1}(a))^*r\,, \tag{18}$$

for any $r \in L^2[0,L]$*.*

Proof. The linearity of $A(a)$ is evident from its definition. We will now demonstrate that $A(a)$ is bijective. For $a \in Ad$, we can argue as in Lemma 1, to prove the existence and uniqueness of a solution $u \in H_0^1[0,L]$ of $-au'' = w$ with homogeneous boundary conditions, for any $w \in L^2[0,L]$. The elliptic regularity (e.g. (Brezis, 2010)) imply that $u \in H^2[0,L]$. Therefore, the bijectivety is established. Consequently, there exists a inverse for $A(a)$ that is a linear operator $A^{-1}(a)$.

Moreover

$$\|A(a)u\|_{L^2[0,L]} = \|-au''\|_{L^2[0,L]} \leq \overline{a}\|u''\|_{L^2[0,L]} \leq \overline{a}\|u\|_{H^2[0,L]}\,. \tag{19}$$

that implies that $A(a)$ is bounded. The open map theorem (Brezis, 2010) implies that $A^{-1}(a)$ is bounded. This verifies the claims of item i)-ii).

The proof of item iii) is a direct consequence of the bounded linear operator theory, as seen in (Brezis, 2010), and items i) and ii). Furthermore,

$$\langle A^*(a)v, \varphi\rangle_{L^2} = \langle v, A(a)\varphi\rangle_{L^2} = \langle v, -a\varphi''\rangle_{L^2} = \langle -(av)'', \varphi\rangle_{L^2}$$

for any $\varphi \in C_0^\infty[0,L]$. Therefore, the fact that $C_0^\infty[0,L]$ is dense in $L^2[0,L]$ implies that $A^*(a)v = -(av)''$.

The proof of items iii) and iv) is next. Let $F(a+h) = \tilde{u}$ and $F(a) = u$ with $\tilde{a} := a+h \in Ad$. The linearity of (1) implies that the difference $F(a+h)$ and $F(a)$ satisfies

$$-a(\tilde{u}'' - u'') = h\tilde{u}'' \,, \tag{20}$$

with homogeneous boundary conditions. It follows from $\tilde{a} \in Ad$ that $\tilde{u}'' = \tilde{a}^{-1}M \in L^2[0,L]$. Then, arguing as in Lemma 1, we get the existence and uniqueness of a solution $U = U(h) \in H^2[0,L] \subset L^2[0,L]$ of (20). Set $F'(a)h := U$. As a consequence, $F'(a)$ is a linear operator in h, for $a+h \in Ad$. Furthermore, he elliptic regularity theory, e.g., (Brezis, 2010) implies in

$$\|U\|_{L^2[0,L]} \leq \underline{a}^{-1} \|M\|_{L^2[0,L]} \|h\|_{L^2[0,L]} \,.$$

As a result, $F'(a)$ can be extended as a bounded liner operator for any $h \in L^2[0,L]$ that satisfies $\|F'(a)\| \leq \underline{a}^{-1}\|M\|_{L^2[0,L]} := \gamma^{-1}$,.

It can be deduced from items i) and ii) that $F'(a)h = U = U(h) = A(a)^{-1}(hu'')$. To prove that $F'(a)$ is the Fréchet derivative of F, let $W = F(a+h) - F(a) - F'(a)h$ for any $h \in L^2[0,L]$ such that $a+h \in Ad$. The linearity implies that W satisfies

$$-aW'' = 0,$$

with homogeneous boundary conditions. As a consequence of the maximum principle (Brezis, 2010) we get $W(x) = 0 \ , \forall x \in [0,L]$, for any h. Therefore,

$$\lim_{\|h\| \to 0} \frac{\|F(a+h) - F(a) - F'(a)h\|}{\|h\|} = 0 \,,$$

concluding item iii).

Item iii) implies the existence of a bounded linear operator $F'(a)^*$ (adjoint of $F'(a)$) satisfying

$$\langle F'(a)^* r, h \rangle_{L^2} = \langle r, F'(a)h \rangle_{L^2} , \quad h \in L^2[0,L]. \tag{21}$$

From (16), it follows that

$$\langle r, F'(a)h\rangle_{L^2} = \langle r, A^{-1}(a)[hu'']\rangle_{L^2} = \langle u''(a)(A^{-1}(a))^{*}r, h\rangle_{L^2}, \tag{22}$$

that concludes item iv). □

3 THE SMOOTHNESS OF THE IDENTIFICATION BY CLASSICAL REGULARIZATION METHODS

In this section, we discuss the smoothness of the reconstructed stiffness coefficient $a_\alpha^\delta(x)$ of a beam given by classical regularization methods (Kaltenbacher et al., 2008; Kirsh, 2011).

In the Tikhonov type regularization, $a_\alpha^\delta(x)$ is obtained as

$$a_\alpha^\delta(x) \in \arg min \|F(a) - u^\delta\|^2_{L^2[0,L]} + \alpha\|a - a_0\|_{L^2[0,L]} \tag{23}$$

where α is the regularization parameter and a_0 represent some a-priori information about the solution of the problem.

In the iterative type regularization, $a_\alpha^\delta(x)$ is obtained as

$$a_\alpha^\delta(x) := a_{k^*}^\delta \tag{24}$$

where k^* is the first index of the iteration

$$a_{k+1}^\delta = a_k^\delta + \gamma_k F'(a_k^\delta)^*(F(a_k^\delta - u^\delta)\,, \tag{25}$$

satisfying the discrepancy principle

$$\|F(a_k^\delta) - u^\delta\|_{L^2[0,L]} \le \tau\delta\,, \tag{26}$$

for some $\tau > 1$. In particular, if

$$\gamma_k := \frac{\|F'(a_k^\delta)^*(F(a_k^\delta - u^\delta)\|^2}{\|F'(a_k^\delta)F'(a_k^\delta)^*(F(a_k^\delta - u^\delta)\|^2}\,, \quad \text{or} \quad \gamma_k = \gamma^{-1}\,, \tag{27}$$

we have the *Steepest Descent* or *Landweber* iteration, respectively, e.g. (Kaltenbacher et al., 2008).

In the following theorem, we show that Tikhonov regularization (23) as well as iterative regularization (25) produce smooth reconstructed coefficients a_α^δ.

Theorem 5. *Let the Assumptions **H1) - H3)** holds. Assume that the a-priori a_0 is smooth. Then, the reconstructed coefficient a_k^δ given by the Tikhonov regularization* (23) *or by iterative regularization* (24) *is smooth.*

Proof. The existence of a minimizer a_α^δ for the Tikhonov functional (23) follows from assumptions, Theorem 3 and the standard extraction of subsequences as in (Kaltenbacher et al., 2008; Kirsh, 2011). The first order optimality conditions implies that

$$a_\alpha^\delta \in a_0 + \frac{1}{2\alpha} F'(a_\alpha^\delta)^*(F(a_\alpha^\delta) - u^\delta) \tag{28}$$

It follows from (28) or (25) that the regularizing solutions a_α^δ are as smooth as $F'(a_\alpha^\delta)^*(F(a_\alpha^\delta) - u^\delta)$. Therefore, its follows from Propositon 4 iv) that a_α^δ is at least $H^2[0, L]$. □

As a consequence of Theorem 5, any regularizing solution a_α^δ obtained by the classical regularization approaches is not suitable for identifying the piecewise constant rigidity coefficient $a(x)$ as in (4).

4 A LEVEL-SET ANSATZ

Given the results in Theorem 5, in this section we analyze a standard level set (SLS) approach[1] (Cezaro et al., 2009,1) to parameterize the problem of identifying the stiffness coefficient $a(x)$, as in (4). The (SLS) consists of obtaining a level set function $\phi : [0, L] \to \mathbb{R}$, in $H^1([0, L])$, in such a way that the zero level set $\Gamma_\phi := \{x \in [0, L]\,;\ \phi(x) = 0\}$ coincides with the discontinuities of the stiffness parameter $a(x)$ as in (4) coincide with Γ_ϕ. For this reason, the technique is called a level set (Cezaro et al., 2009,1) and references therein.

[1] The SLS approach is distinct from other level-set approaches presented in Cezaro et al. (2013). Some advantages and disadvantages of level-set approaches were discussed in Cezaro et al. (2013).

Let $H(t) = 1$ if $t > 0$ and $H(t) = 0$ if $t \leq 0$ be the Heaviside function. Then, the stiffness coefficient $a(x)$ as in (4) can be rewritten as

$$a(x) = a_2 + (a_1 - a_2)H(\phi) =: P(\phi). \tag{29}$$

Remark 1. *Notice that* $\Omega_1 = \{x \in [0, L] \,:\, \phi(x) \geq 0\}$, $\Omega_2 = \{x \in [0, L] \,:\, \phi(x) < 0\}$. *Therefore, the operator* P *defined in* (29) *establishes the relationship between the level-sets of* ϕ *and the sets* Ω_1, Ω_2 *and the stiffness coefficient* $a(x)$*, as in* (4)*.*

Thus, the operator P *in* (29) *maps* $H^1[0, L]$ *into the set*

$$\mathcal{V} \;:=\; \{z \in L^\infty([0, L]) \,|\, z = a_2 + (a_1 - a_2)\,\chi_{\Omega_1}\,,\ \textit{for some}\ \ \Omega_1 \subset [0, L]\}\,. \tag{30}$$

It follows from Remark 1 that the problem of identifying the stiffness coefficient $a(x)$ as in (4), can be rewritten as

$$F(P(\phi)) \;=\; u^\delta, \tag{31}$$

where u^δ are the measurements in (3).

4.1 The regularization properties of the (SLS)-approach

Since the inverse problem of identifying $a(x)$ is ill-posed (see Theorem 3), the some characteristic is shared by problem (31). Therefore, we shall propose an regularization strategy for the level-set anzat.

A regularization strategy for the (SLS) approach presented previously for the problem (31), consists of obtaining regularized solutions ϕ^δ_α, for the Tikhonov functional

$$\mathcal{F}_\alpha(\phi) :=\|F(P(\phi)) - u^\delta\|^2_{L^2[0,L]} + \alpha\left(\beta_1|H(\phi)|_{BV} + \beta_2\|\phi - \phi_0\|^2_{H^1([0,L])}\right)\,, \tag{32}$$

where $\alpha > 0$ is the regularization parameter, while the positive constants β_1, β_2 are scaling factors. Furthermore, $|\cdot|_{BV}$, represents the semi-norm in the space of bounded variation functions; see (Evans & Gariepy, 1992). The term in H^1 acts as a regularization for the level set function ϕ in the space H^1. The semi-norm BV penalizes the size of the Hausdorf measure on the boundary of the set $\{x \in [0, L] : \phi(x) > 0\}$.

As the operator P is discontinuous, the functional $\mathcal{F}_\alpha$ does not have a closed graph in $H^1[0,L]$. Therefore, the existence of minimizers for $\mathcal{F}_\alpha$ will be proven, in a generalized sense (see (Cezaro et al., 2009)), as follows: for each $\varepsilon > 0$, consider the continuous approximation of the operator P as $P_\varepsilon(\phi) := a_1 H_\varepsilon(\phi) + a_2(1 - H_\varepsilon(\phi))$, where $H_\varepsilon(t) = 1 + t/\varepsilon$ for $t \in [-\varepsilon, 0]$, $H_\varepsilon(t) = H(t)$ for $t \in \mathbb{R}\setminus[-\varepsilon, 0]$. The pair $(z,\phi) \in L^\infty[0,L] \times H^1[0,L]$ is said to be *admissible* when there is a sequence of functions $\{\phi_k\}$ in $H^1[0,L]$ and a sequence of positive numbers $\varepsilon_k \to 0$, satisfying

$$\lim_{k\to\infty} \|\phi_k - \phi\|_{L^2[0,L]} = 0\,,\ \lim_{k\to\infty} \|H_{\varepsilon_k}(\phi_k) - z\|_{L^1[0,L]} = 0\,. \tag{33}$$

A *generalized minimum* for $\mathcal{F}_\alpha$ defined in (32) will be considered as any admissible pair (z,ϕ) minimizing

$$\mathcal{G}_\alpha(z,\phi) := \|F(Q(z)) - u^\delta\|^2_{L^2[0,L]} + \alpha\left(\inf\left\{\liminf_{k\to\infty}\left(\beta_1|H_{\varepsilon_k}(\phi_k)|_{BV} + \beta_2\|\phi_k - \phi_0\|_{H^1[0,L]}\right)\right\}\right)\,, \tag{34}$$

over the set of admissible pairs, where $Q : L^\infty[0,L] \ni z \mapsto a_1 z + a_2(1-z) \in L^\infty[0,L]$.

For the convergence analysis that follows, we need to set an extra assumption: Assumption **(A1)**: There is a^* piecewise constant, solution of (6) and there is $\phi^* \in H^1[0,L]$ such that $P(\phi^*) = a^*$ with $|\nabla\phi^*| \neq 0$ in a neighborhood of $\{\phi^* = 0\}$, such that $H(\phi^*) = z \in \mathcal{V}$.

Theorem 6 (Existence of generalized minimizers, Convergence and Stability)**.** *Consider hypothesis A1) true. Then:*

i) The functional $\mathcal{G}_\alpha$ in (32) *has minimizers in the set of admissible pairs.*

ii) [Convergence] Assume that you have exact data $u^\delta = u$. For each $\alpha > 0$ denote by (z_α, ϕ_α) a minimizer of $\mathcal{G}_\alpha$ in the set of admissible pairs. Then, for any sequence of positive numbers $\{\alpha_k\} \to 0$, there exists a subsequence (which for simplicity will be denoted by $\{\alpha_k\}$), such that $(z_{\alpha_k}, \phi_{\alpha_k}) \to (\hat{z}, \hat{\phi})$ in $L^1[0,L] \times L^2[0,L]$, with $(\hat{z}, \hat{\phi})$. Still, $(\hat{z}, \hat{\phi})$ is a solution of (31)*.*

iii) [Stability] Let $\alpha = \alpha(\delta)$ satisfy $\lim_{\delta\to 0}\alpha(\delta) = 0$ and $\lim_{\delta\to 0}\delta^2\alpha(\delta)^{-1} = 0$. For any sequence of positive numbers $\{\delta_k\} \to 0$, denote by $\{u^{\delta_k}\} \in L^2[0,L]$ the noisy measurements *satisfying* (3)*. Then, there is a subsequence, which is for simplicity denoted by* $\{\delta_k\}$*, and its corresponding* $\{\alpha_k := \alpha(\delta_k)\}$ *such that* $(z_{\alpha_k}, \phi_{\alpha_k})$ *converges in* $L^1[0,L] \times L^2[0,L]$ *for a solution of* (31)*.*

Proof. It follows from Theorem 2 ii) that the operator F in (6) is continuous in the topology of $L^1[0,L]$. Therefore, all the hypotheses required in (Cezaro et al., 2009, Theorem 6, 8, 9) are satisfied. Therefore, the proof follows as a particular case of the results obtained in (Cezaro et al., 2009, Theorem 6, 8, 9). □

4.2 Numerical approximation

Table 1 – Level-Set Algorithm

Step 1: Evaluate the residue $r_k := F(P_\varepsilon(\phi_k)) - u^\delta$ where $F(P_\varepsilon(\phi_k)) = u_\varepsilon$ solution of $P_\varepsilon(\phi_k))u''_\varepsilon = M$ with boundary conditions (2)
Step 2: Evaluate $F'(P_\varepsilon(\phi_k))^* r_k$ For that: Step 2.1: Solve $-(P_\varepsilon(\phi_k)v_k)'' = r_k$ with conditions $v'(0) = v'(1) = 0$ Step 2.2: Calculate $\frac{M}{P_\varepsilon(\phi_k)} v_k$ where v_k is the solution of Step 2.1
Step 3: Calculate $R_\varepsilon(\phi_k)$ given by the right side of (36) for $\phi = \phi_k$
Step 4: Evaluate $\delta\phi_k$ solution of $\alpha\beta_1(I - \Delta)\delta\phi_k = -R_\varepsilon(\phi_k)$ with $(\delta\phi)_V = 0$ and $R_\varepsilon(\phi_k)$ calculated in Step 3 according to (36)
Step 5: Update $\phi_{k+1} = \phi_k + \delta\phi_k$

Source: the authors (2024)

As noted in Cezaro et al. (2009), obtaining generalized minima for functional $\mathcal{F}_\alpha$ is impractical. Alternatively, the strategy is to obtain minimizers for the functional

$$\mathcal{G}_{\varepsilon,\alpha}(\phi) := \|F(P_\varepsilon(\phi)) - u^\delta\|^2_{L^2[0,L]} + \alpha\left(\beta_1 |H_\varepsilon(\phi)|_{BV} + \beta_2\|\phi - \phi_0\|^2_{H^1(\Omega)}\right) \quad (35)$$

which has minimizers in $H^1[0,L]$, which approximate the minimizers of the functional $\mathcal{F}_\alpha$ when $\varepsilon \to 0$. In fact, Theorem 2 ii) and hypothesis A1) guarantee that the results presented in (Cezaro et al., 2009, Lemma 10, Theorem 11) can be replicated for the problem studied in this work, of which the above statement follows. Thus, the numerical method will be based on the optimality conditions for the $\mathcal{G}_{\varepsilon,\alpha}$ functional given by[2]

$$\beta_2\alpha\left[(\phi - \phi_0)'' - (\phi - \phi_0)\right] = (a_1 - a_2)H'_\varepsilon(\phi)F'(P_\varepsilon(\phi))^*(F(P_\varepsilon(\phi)) - u^\delta) + 1/2\beta_1\alpha\varepsilon^{-2} sign(H'_\varepsilon(\phi)), \quad (36)$$

[2]Note that $H'_\varepsilon(t) = 1/\varepsilon$, $t \in (-\varepsilon, 0)$, $H''_\varepsilon(t) = -1/\varepsilon^2$, $t \in (-\varepsilon, 0)$ and zero otherwise. sign represents the sign function.

with boundary conditions $(\phi - \phi_0)'(0) = (\phi - \phi_0)'(L) = 0$.

The numerical implementation to obtain the coefficient (4) is given by the iterative algorithm presented in Table (1). The calculations used to obtain the adjoint of the Fréchet derivative of the operator F are given in Proposition 4.

5 SIMULATED RESULTS

In this subsection, to support the theoretical results obtained previously, some numerical simulations of the algorithm 1 will be presented to identify a piecewise constant stiffness coefficient $a(x)$ given by (4). The measured data simulated correspond to a loading force of $f(x) = 1$ applied to a beam of length $L = 1$. For all cases, the data were simulated in the manner $u^\delta = u(x) + \xi$, where ξ is a uniformly distributed random variable with values between $[-1, 1]$ and $u(x) = u(a^*(x))$ is the solution of (1), where $a^*(x)$ is the coefficient to be identified, given by

$$a^*(x) \begin{cases} 0.5 & \text{if} \quad x \in [0, 1/2] \\ 1.0 & \text{if} \quad x \in]1/2, 1] \end{cases} . \tag{37}$$

In all simulations presented, $\varepsilon = 1/100$ and the initial guess is given by $\phi_0 = x(x - 1/4)$.

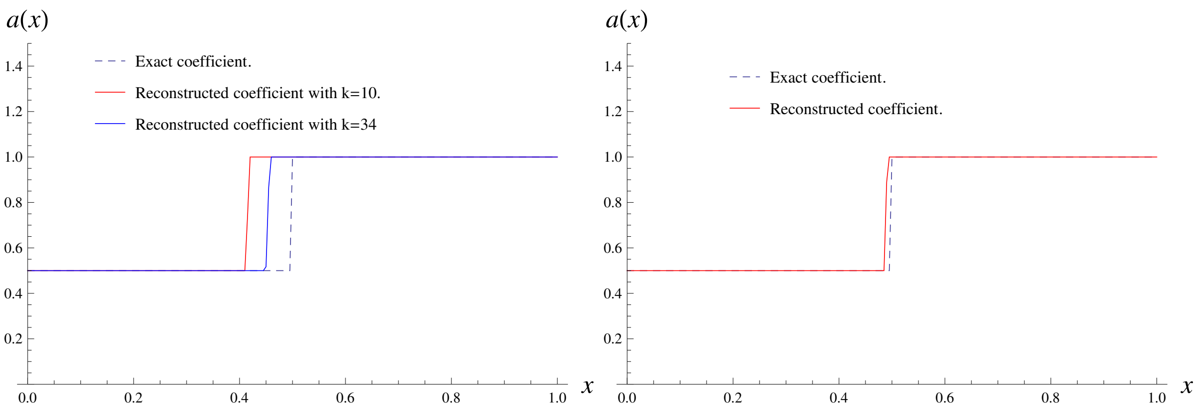
It is worth mentioning that at least two ODE's are solved in each iteration step of the algorithm 1. The numerical solution was obtained using the finite difference method, with equal distributed points $x_i = i/N$, $i = 0, 1, 2, \ldots, N$, and $N = 200$ points.

Figures 1(a)-(b) present the recovered coefficient $a_k^\delta = P_\varepsilon(\phi_k)$ according to (29) for the noise level $\delta = 0.001$, at different stages of the iteration. It is possible to observe that the coefficient iterated by the algorithm 1 approaches the exact solution.

Figure 2 presents the recovered coefficient for the noise level of $\delta = 0.01$ at different points of the k iteration. Once again, it is possible to observe that the iteration of the algorithm produces approximate solutions that converge to solve the problem.

The numerical simulations presented in Figures 1 and 2 demonstrate that the approximate iterative solution $a_k^\delta = P_\varepsilon(\phi_k)$ is stable and convergent for the constant stiffness coefficient piecewise $a^*(x)$, consistent with the theoretical results presented in Theorem 6.

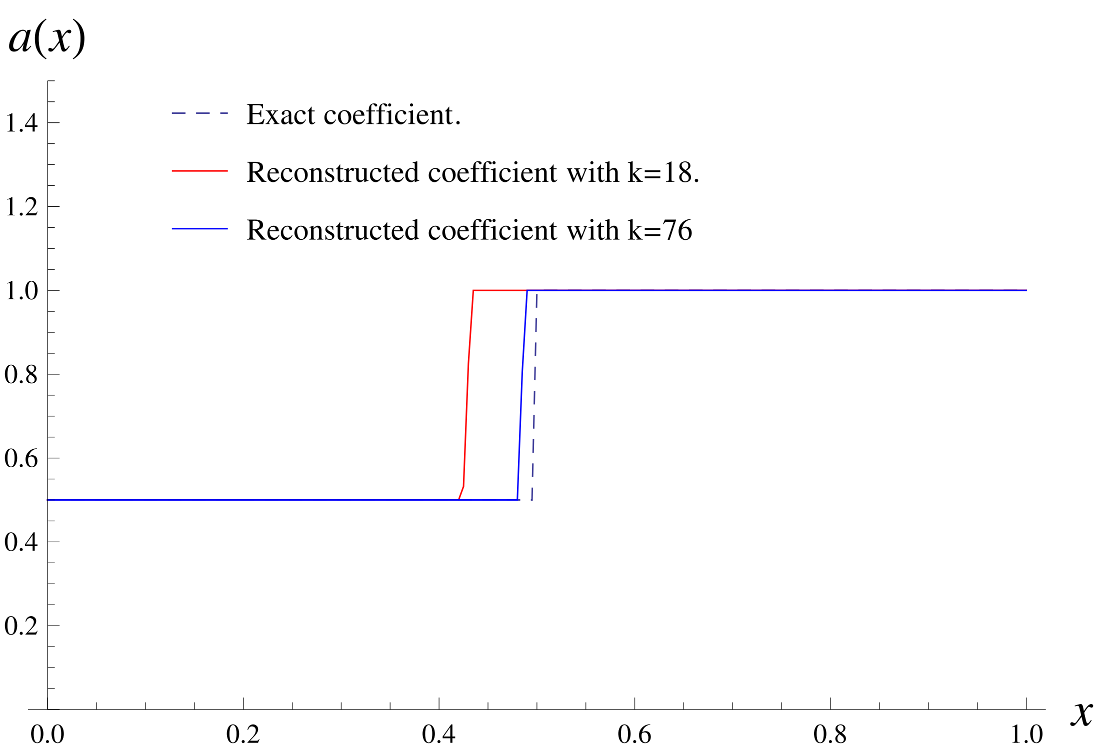
Figure 1 – Reconstructed stiffness coefficient $a_k^\delta = P_\varepsilon(\phi_k)$ for the noise level of $\delta = 0.001$



(a) Coefficient $a_k^\delta = P_\varepsilon(\phi_k)$ at distinct iteration index k. (b) Coefficient $a_k^\delta = P_\varepsilon(\phi)$ after $k = 85$ iterations.

Caption: The (a) figure shows the reconstructed coefficient for $k = 10$ and $k = 34$ iterations and (b) shows the coefficient after $k = 85$ iterations
Source: the authors (2024)

Figure 2 – Reconstructed stiffness coefficient $a_k^\delta = P_\varepsilon(\phi_k)$ for the noise level of $\delta = 0.01$ at distinct iteration index k



Caption: The graph shows the reconstructed coefficient for $k = 18$ and $k = 76$ iterations

6 CONCLUSIONS

In this work, the inverse problem of identifying the piecewise constant coefficient in the static Euler-Bernoulli equation was studied. It proved to be possible to use level set methods as a regularization method for the problem through the continuity of the operator (6) parameter-to-measurements in the topology of L^1. Furthermore, the necessary properties were presented to obtain regularized solutions of the level set function in a stable and convergent way. This work was concluded by presenting numerical simulations with the objective of identifying the constant coefficient by parts at different noise levels using the proposed level set method, corroborating the theoretical results previously constructed.

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