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Mathematics

Analysis of solutions and specific properties involving the circle circumscribed to the triangle

Análise de soluções e propriedades específicas envolvendo o círculo circunscrito ao triângulo

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ABSTRACT

This work involves algebraic and geometric analysis of specific plane figures and seeks to analyze some unusual relationships between the circumscribed circle and the triangle, in addition to establishing conditions and intervals of their existence. We obtain rational curves that express, for example, maximum area values or asymptotic points. Also, through computer simulations, we visualize the behavior of these algebraic relationships, establishing a bridge between the theoretical analysis developed and experimental practice.

Keywords: Existence; geometrical quantities; polynomial equations; rational solutions

RESUMO

Este trabalho envolve análise algébrica e geométrica de figuras planas específicas, busca analisar algumas relações incomuns entre o círculo circunscrito e o triângulo, além de estabelecer condições e intervalos de sua existência. Obtemos curvas racionais que expressam, por exemplo, valores máximos de área ou pontos assintóticos. Também, por meio de simulações computacionais, visualizamos o comportamento dessas relações algébricas, estabelecendo uma ponte entre a análise teórica desenvolvida e a prática experimental.

Palavras-chave: Existência; grandezas geométricas; equações polinomiais; soluções racionais

1 Introduction

The circle circumscribed to polygons has been the subject of study in different areas of exact sciences with diverse objectives and applications. Byerly (1909) works with Poncelet's Theorem, which deals with the problem of a polygon inscribed in a circle and circumscribed by another circle, detailing two cases of quadrilaterals and a third generalized case of these. Hayden (1936) presents the circumscribed, inscribed and escribed circles of a triangle in trilinear coordinates, comparing them for teaching purposes. Wardhill (1947) investigates the equation of the circumscribed circle to the triangle in areal coordinates using the triangle sides as parameters. Pinkham (1996) studies the importance and potential of the computation, considering approximations with upper and lower bounds in the analysis of a sequence of circumscribed circles in regular polygons. Jywe et al. (1999) investigate the geometric error between an ideal object and a sample, based on the smallest circumscribed circle and the largest inscribed circle to an object, to obtain the smallest circular zone. To do this, the vertices of an iterative triangle are used as control points, as well as circumscribed and inscribed circles. Kolar–Super et al. (2010) prove the existence of three triangles that touch the circumscribed circle and the nine-point circle of a triangle, presenting relationships between these circles and elements of the triangle. Metaxas and Karagiannidou (2010) approach the teaching of geometry using a computational environment, bringing conditions for the existence of a triangle with its inscribed and circumscribed circles, exploring the relationship between its radii and a relationship between the semiperimeter with the radii and a segment (given by a triangle vertex and the center of the inscribed circle). Avksent'ev (2012) brings metric properties of Poncelet polygonal lines, explaining formulas for inscribed and circumscribed polygons with 5, 6 and 8 sides and an algorithm for determining n sides. Garcia (2019) concluded that the centers of circumscribed and inscribed circles to the triangles that are the 3-periodic orbits of an elliptic billiard are ellipses and obtained the canonical equations of these ellipses. Parolin and Darós (2024) approach, through roots of polynomial functions and Lagrange multipliers, different relationships of geometric quantities of the circle inscribed in a triangle. They also present and analyze the behavior of specific trigonometric solutions of these relationships and dynamize the study for other geometric shapes. In Parolin et al. (2024), they explore mean functions of the semicircle related to the triangle, in addition to developing Dynamic Geometry objects.

In this article we continue studies involving geometric objects and their quantities relationships. We work with the circle circumscribed to the triangle, looking to describe curves that relate the perimeter and the triangle area with the radius and area of the circumscribed circle. In this context, conditions and existence intervals for such curves are obtained, given by rational solutions and which present characteristics, let's say curious, with regard to curvature, points of maximum area of the triangle and asymptotic points.

2 Main Results

This work aims to analyze the existence of relations between geometric quantities of the triangle and its circumscribed circle, comparing from the equation

$$A = \frac{a \cdot b \cdot c}{4r}, \tag{1}$$

the triangle area (A) and the radius (r) of these geometric shapes. Note that equation (1), although seemingly quite simple, it hides important relations of geometry and with the use of the well-known Heron's formula (Oliver (1993))

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \tag{2}$$

where the semiperimeter $s = \frac{a+b+c}{2}$.

So, replacing this semiperimeter s in equation (2) we have

$$A = \frac{1}{4}\sqrt{-c^4 + 2c^2(a^2+b^2) - (a^2-b^2)^2}. \tag{3}$$

Analyzing the equations (1) and (3) and isolating the variable r, we obtain the relation of the radius to the sides of the triangle,

$$r = \frac{a \cdot b \cdot c}{\sqrt{-c^4 + 2c^2(a^2+b^2) - (a^2-b^2)^2}}. \tag{4}$$

Lemma 1: Fixed $a > b$, consider $f(c) = -c^4 + 2(a^2+b^2)c^2 - (a^2-b^2)^2$. So, f is strictly positive in the intervals $(-a-b, -a+b)$ and $(a-b, a+b)$.

proof. In fact, initially we consider the denominator of equation (4)

$$-c^4 + 2c^2(a^2 + b^2) - (a^2 - b^2)^2 = 0. \tag{5}$$

Then, making the variable change $c^2 = x$ we have

$$-x^2 + 2x(a^2 + b^2) - (a^2 - b^2)^2 = 0. \tag{6}$$

Note that with the study of the discriminant signal

$$\Delta = 4[(a^2 + b^2)^2 - (a^2 - b^2)^2] = 16a^2b^2 \tag{7}$$

we obtain $x_1 = (a - b)^2$ and $x_2 = (a + b)^2$, and consequently,

$$c_1 = -a - b, c_2 = -a + b, c_3 = a - b, c_4 = a + b. \tag{8}$$

Now let's look for critical points of the polynomial function $f(c)$. So, calculating the first derivative in terms of c, we obtain

$$f^{'}(c) = -4c^3 + 4(a^2 + b^2)c$$

and therefore, $f^{'}(c) = 0$ if, and only if

$$c = 0 \text{ or } c = \pm\sqrt{a^2 + b^2}.$$

Applying the second derivative test in terms of c, namely $f^{''}(c) = -12c^2 + 4(a^2 + b^2)$, we conclude that

$$f^{''}(-\sqrt{a^2 + b^2}) < 0, \ \ f^{''}(0) > 0 \text{ and } f^{''}(\sqrt{a^2 + b^2}) < 0.$$

In this way, $c = \pm\sqrt{a^2 + b^2}$ are maximum points and $c = 0$ is a local minimal point of this function f.

Therefore, analyzing the information obtained by the first and second derivative tests, we have that f is positive in the intervals (c_1, c_2) and (c_3, c_4). ■

By Lemma 1 we observed that the interval of existence of the relation between the radius and the sides of the triangle (4), is given by $(|a-b|, a+b)$.

So, considering this interval of existence, we have

$$\frac{dr}{dc} = \frac{abc^4 - ab(a^2-b^2)^2}{[-c^4 + 2c^2(a^2+b^2) - (a^2-b^2)^2]^{\frac{3}{2}}}.$$

From there, analyzing its critical points, $\frac{dr}{dc}$ does not exist in the boundary of $(|a-b|, a+b)$. It is also observed that if $a \neq b$,

$$\lim_{c \to (a+b)^-} r(c) = +\infty \quad \text{and} \quad \lim_{c \to |a-b|^+} r(c) = +\infty,$$

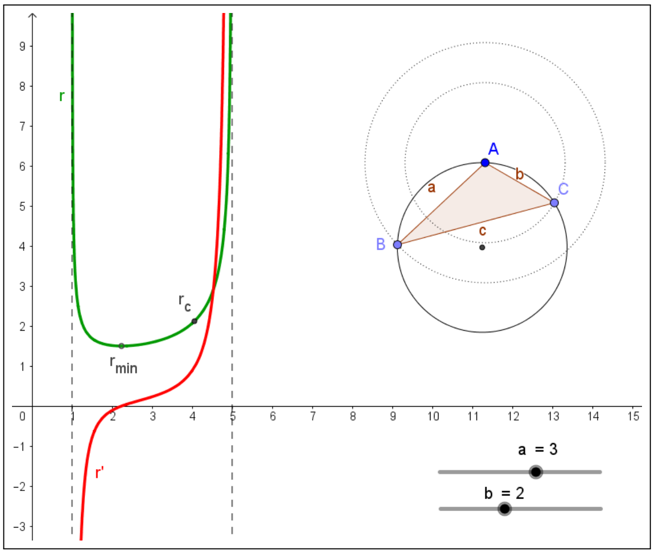
and if $a = b$,

$$\lim_{c \to 0^+} r(c) = \frac{a}{2}.$$

This characterizes the upper boundary of the interval as a vertical asymptote and, for the $a \neq b$ case, we also have the lower boundary as a vertical asymptote, Figure 1.

Furthermore, $\frac{dr}{dc} = 0$ when $c = \sqrt{|a^2-b^2|}$ for $a \neq b$. Yet, this is the minimum point with minimum value $r_{\text{min}}(c) = \frac{a}{2}$ for $a > b$ or $r_{\text{min}}(c) = \frac{b}{2}$ for $a < b$, because $\frac{dr}{dc} > 0$, for $c > \sqrt{|a^2-b^2|}$ and $\frac{dr}{dc} < 0$, for $c < \sqrt{|a^2-b^2|}$.

Figure 1 – Radius as a function in terms of the variable side c, $r(c)$ and $r'(c)$



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In the other hand, working with the equations (1) and (3) in terms of variable c, we have that

$$c^4 + c^2\left[\left(\frac{ab}{r}\right)^2 - 2(a^2+b^2)\right] + (a^2-b^2)^2 = 0. \tag{9}$$

So, making the change of variable $x = c^2$, we rewrite the equation above as

$$x^2 + x\left[\left(\frac{ab}{r}\right)^2 - 2(a^2+b^2)\right] + (a^2-b^2)^2 = 0,$$

which has the following solutions to the equation (9)

$$c = \pm\frac{\sqrt{2}}{2r}\sqrt{2r^2(a^2+b^2) - a^2b^2 \pm ab\sqrt{(a^2-4r^2)(b^2-4r^2)}}.$$

Considering the variables c and r positive, we highlight in the problem discussed, the solutions

$$c_1 = \frac{\sqrt{2}}{2r}\sqrt{2r^2(a^2+b^2) - a^2b^2 + ab\sqrt{(a^2-4r^2)(b^2-4r^2)}} \tag{10}$$

and

$$c_2 = \frac{\sqrt{2}}{2r}\sqrt{2r^2(a^2+b^2) - a^2b^2 - ab\sqrt{(a^2-4r^2)(b^2-4r^2)}} \tag{11}$$

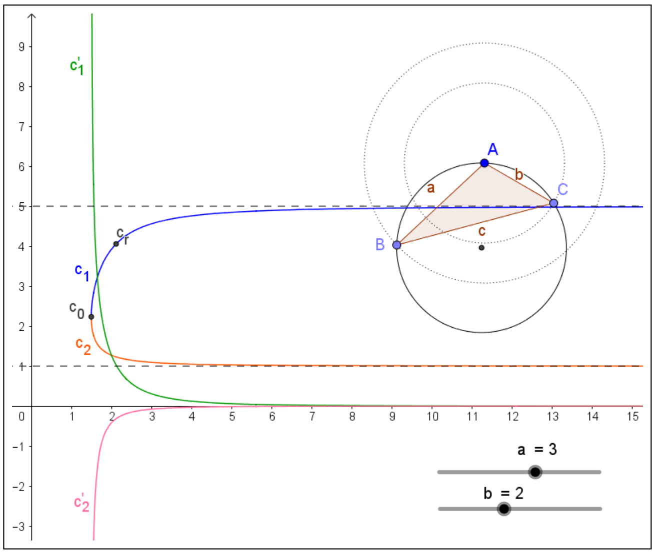
for $a \neq b$. In particular, when $a = b$ we get a simpler solution

$$c = \frac{a}{r}\sqrt{4r^2 - a^2}. \tag{12}$$

After some algebraic analysis of the expressions (10) and (11), we verified the existence of c_1 and c_2 for $r \geq \frac{a}{2}$, whenever $a > b$ or, when $a < b$, for $r \geq \frac{b}{2}$, Figure 2. This result is simply extended to the case where $a = b$. From now on we will always assume $a > b$.

Also, note that $c_2 \leq c_0 \leq c_1$ with a single point of intersection $c_0 = \sqrt{a^2 - b^2}$ for this solution, that occurs in $r = \frac{a}{2}$. Furthermore, we verified to c_0 the occurrence of a right triangle with hypotenuse in a. For $c_1 > c_0$ we have an acute angle in $\hat{C}$ while for $c_2 < c_0$ we get an obtuse angle in $\hat{C}$.

Figure 2 – Geometric Representation of an Inscribed Right Triangle and Curves c_1 and c_2.



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For simplicity, calculating implicit differentiation of equation (9), we have

$$\frac{dc}{dr} = \frac{a^2b^2c}{2r^3(c^2-a^2-b^2)+a^2b^2r},$$

and substituting c_1 and c_2, it follows that

$$\frac{dc_1}{dr} = \frac{ab\sqrt{2r^2(a^2+b^2)-a^2b^2+ab\sqrt{(a^2-4r^2)(b^2-4r^2)}}}{r^2\sqrt{2(a^2-4r^2)(b^2-4r^2)}}$$

and

$$\frac{dc_2}{dr} = -\frac{ab\sqrt{2r^2(a^2+b^2)-a^2b^2-ab\sqrt{(a^2-4r^2)(b^2-4r^2)}}}{r^2\sqrt{2(a^2-4r^2)(b^2-4r^2)}}.$$

Note that these equations do not exist for $r = \frac{a}{2}$. In particular, for $a = b$ we have from (12) the following equation $\frac{dc}{dr} = \frac{a^2}{r^2\sqrt{4r^2-a}}$. This equation also does not exist for the same value of r. Such derivatives never cancel each other and are positive and negative, respectively.

Furthermore,

$$\lim_{r\to+\infty} c_1(r) = a+b \quad \text{and} \quad \lim_{r\to+\infty} c_2(r) = a-b,$$

characterizing a horizontal asymptote for each solution.

Another possible analysis to be carried out with equations (1) and (3) refers to relating the triangle area to the geometric quantities of the circumscribed circle.

Therefore, relating (2) with solutions c_1 and c_2 obtained in (10) and (11), we can write the areas of the triangle as a function of the radius r of the circle, respectively, by

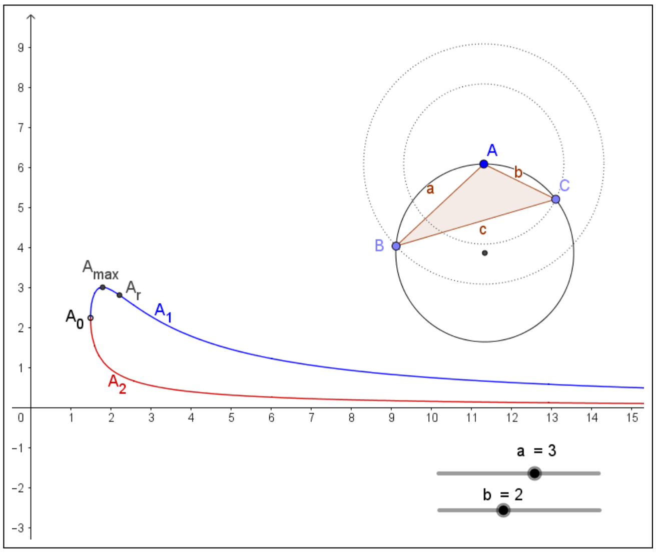
$$\begin{aligned} A_1 &= \tfrac{\sqrt{2}ab}{8r^2}\sqrt{2r^2(a^2+b^2)-a^2b^2+ab\sqrt{(a^2-4r^2)(b^2-4r^2)}} \quad \text{and} \\ A_2 &= \tfrac{\sqrt{2}ab}{8r^2}\sqrt{2r^2(a^2+b^2)-a^2b^2-ab\sqrt{(a^2-4r^2)(b^2-4r^2)}}. \end{aligned} \tag{13}$$

As expected, there is an intersection point

$$A_0 = \frac{b\sqrt{a^2-b^2}}{2}$$

of these equations, where $A_2 \leq A_0 \leq A_1$, obtained from r_{min}, Figure 3. Furthermore, the maximum area of the inscribed triangle $A_{\text{max}} = \frac{ab}{2}$ occurs when there is a right angle at $\hat{A}$ with radius of the circle $r = \frac{\sqrt{a^2+b^2}}{2}$, since the hypotenuse of an inscribed triangle coincides with the diameter of the circle.

Figure 3 – Triangle area as a function of the radius variable r, $A(r)$



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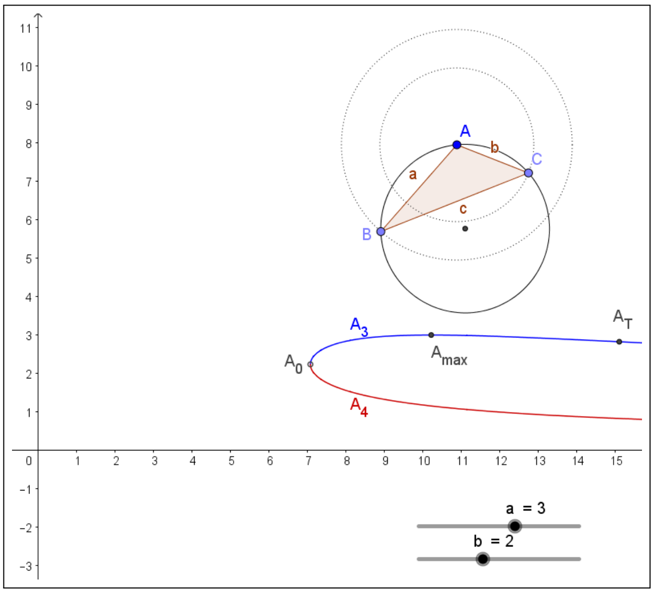
Also, we could think about relating the triangle area (A_T) and the area of the circumscribed circle (A_C). To do so, rewriting the equations in (13) as a function of the area of the circle, we obtain

$$A_3 = \frac{\sqrt{2}\pi ab}{8A_C}\sqrt{2\frac{A_C}{\pi}(a^2+b^2) - a^2b^2 + ab\sqrt{(a^2-4\frac{A_C}{\pi})(b^2-4\frac{A_C}{\pi})}} \quad \text{and}$$
$$A_4 = \frac{\sqrt{2}\pi ab}{8A_C}\sqrt{2\frac{A_C}{\pi}(a^2+b^2) - a^2b^2 - ab\sqrt{(a^2-4\frac{A_C}{\pi})(b^2-4\frac{A_C}{\pi})}}. \tag{14}$$

with the maximum area of the inscribed triangle (A_{max}), occurring in the right triangle with right angle at $\hat{A}$ and area of the circle $A_C = \frac{\pi(a^2+b^2)}{4}$. In the same way, there is the point of intersection A_0 with $A_4 \leq A_0 \leq A_3$, obtained from the smallest area of the circle $A_C = \frac{a^2\pi}{4}$, Figure 4.

Figure 4 – Triangle area as a function of the area of the circumscribed circle $A_3(A_C)$ and $A_4(A_C)$



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On the other hand, looking for another association between these same quantities, now with A_C as a function of A_T, we can rewrite the equation (2) as being

$$16A_T^2 = (a+b+c)(b+c-a)(a+c-b)(a+b-c)$$

or, even, as a quartic equation in the parameter c,

$$c^4 - 2(a^2+b^2)c^2 + (a^2-b^2)^2 + 16A_T^2 = 0.$$

Solving this last equation, with the change of variable $x = c^2$, it is possible to obtain four solutions. Considering the nature of the problem, in which the parameter c is positive, we have as solutions

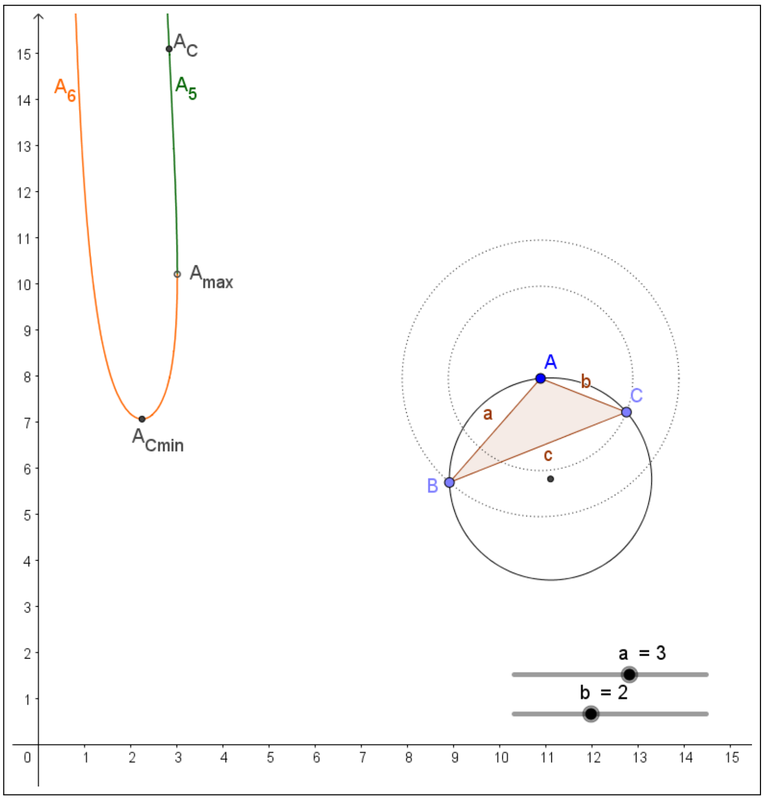
$$\begin{aligned} c_3 &= \sqrt{a^2 + b^2 + 2\sqrt{a^2b^2 - 4A_T^2}} \quad \text{and} \\ c_4 &= \sqrt{a^2 + b^2 + 2\sqrt{a^2b^2 - 4A_T^2}}. \end{aligned} \tag{15}$$

Remembering the equation (1), in addition to the equations (15), we obtain the relationships between the areas of the circle

$$\begin{aligned} A_5 &= \tfrac{a^2b^2\pi}{16A_T^2}\left(a^2 + b^2 + 2\sqrt{a^2b^2 - 4A_T^2}\right) \quad \text{and} \\ A_6 &= \tfrac{a^2b^2\pi}{16A_T^2}\left(a^2 + b^2 - 2\sqrt{a^2b^2 - 4A_T^2}\right). \end{aligned} \tag{16}$$

Figure 5 shows the curves (16) and their properties of maximum triangle area and minimum circumference area, intersection points and geometry of the proposed problem, as already explained.

Figure 5 – Area of the circumscribed circle as a function of the triangle area $A_5(A_T)$ and $A_6(A_T)$



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3 Conclusion

In the search for relations between the triangle area with two fixed sides and the radius of the circumscribed circle, we found rational solutions, which arose from the solution of a fourth degree polynomial equation relating two well-known equations of the triangle area.

After algebraic and geometric analysis of the relationships involving the circumscribed circle and the triangle, we observed conditions of existence for the radius function in terms of the one variable side length in the triangle, $r(c)$. The inverse relationship was also explored, $c(r)$. Minimum points and values were checked in order to understand the behavior of these curves. Furthermore, when relating the triangle area with the quantities of the circumscribed circle, two rational curves are described that complement each other and describe a maximum area when there is a right angle and a certain value for the radius.

Finally, when constructing expressions between the triangle areas and the circle, interesting curves are obtained which, under certain conditions, express the existence of a right triangle and maximum and minimum areas.

For future studies it is possible to think about leaving two sides of the triangle free or even addressing relationships between three-dimensional objects such as pyramids inscribed in the sphere.

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