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Mathematics

Geometry of a navigation problem: the $\lambda-$Funk Finsler Metrics

Geometria de um problema de navegação: a $\lambda-$Funk métrica de Finsler

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ABSTRACT

We investigate the travel time in a navigation problem from a geometric perspective, with respect to a new class of Finsler metrics. We present the $\lambda-$Funk Finsler Metrics. The setting involves an open disk centered at the origin, representing a circular lake perturbed by a symmetric wind flow proportional to the distance from the origin with proportionality factor λ. The Randers metric, which is an important Finsler metric, derived from this physical problem, generalizes the well-known Euclidean metric ($\lambda = 0$) on the Cartesian plane and the Funk metric on the unit disk ($\lambda = 1$). We obtain the formula for distance, or travel time, from point to point, and the circumference equations. In addition, we obtain the distance formulas from point to line and vice versa.

Keywords: Navigation problem; $\lambda-$Funk metric; Finsler metric

RESUMO

Estudamos o tempo de viagem em um problema de navegação desde o ponto de vista geométrico, com respeito a uma nova classe de métricas de Finsler. Apresentamos as métricas $\lambda-$Funk. O problema envolve um disco aberto do plano Euclidiano, que representa um lago circular e é perturbado por um fluxo de vento simétrico e proporcional à sua distância a partir da origem com fator de proporcionalidade λ. A métrica de Randers, que é uma importante métrica Finsler, obtida deste problema físico generaliza as já conhecidas métrica Euclidiana sobre o plano cartesiano ($\lambda = 0$) e métrica de Funk sobre o disco unitário ($\lambda = 1$). Obtemos fórmula de distância, ou tempo de viagem, de ponto a ponto e equação da circunferência. Adicionalmente, obtemos as formulas de distancia de ponto a reta e vice-versa.

Palavras-chave: Problema de navegação; $\lambda-$Funk; Métrica de Finsler

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1 INTRODUCTION

This work addresses travel time in a navigation problem from a geometric perspective, exploring a new class of Finsler metrics called λ-Funk metrics. The problem occurs in an open disk on the Euclidean plane, representing a circular lake, and is perturbed by a symmetric wind flow whose intensity is proportional to the distance from the origin, with a proportionality factor λ. This approach is directly connected to Randers metrics, a fundamental Finsler metric, which in this context generalizes two well-known metrics: the Euclidean metric ($\lambda = 0$) and the Funk metric ($\lambda = 1$).

Finsler metrics generalize Riemannian metrics by allowing the norm of tangent vectors to depend not only on the position in the space but also on the direction. With this, one can think of a Finsler manifold as “a space where the inner product depends not only on where you are but also on in which direction you are looking”. In the context of this work, we introduce the λ-Funk metrics as a particular example of Finsler metrics derived directly from a navigation problem. These metrics provide an explicit formulation for travel time, making them especially useful for describing optimized trajectories in anisotropic environments. This study focuses on the two-dimensional case of the unit disk in $\mathbb{R}^2$, enabling a detailed analysis of distance formulas and associated geometric properties, such as circumferences and point-line distances.

There exist important Finsler metrics; one of them is the Randers metric, defined as the sum of a Riemann metric and a $1-$form. These metrics were first studied by the physicist G. Randers in 1941 from the standard point of general relativity (Randers, 1941). Later on, these metrics were applied to the theory of the electron microscope by R. S. Ingarden in 1957, who first named them Randers metrics. Since then, Randers metrics have been used in many areas like Biology, Ecology, Physics, Seismic Ray Theory, etc.

On the other hand, the Zermelo Navigation problem came to Zermelo’s mind when the airship “Graf Zeppelin” circumnavigated the earth in August 1929. He considered a vector field given in the Euclidean plane that describes the distribution of winds as depending on place and time and treated the question of how an airship or plane, moving at a constant speed against the surrounding air, has to fly in order to reach a given point Q from a given point P in the shortest time possible (Ebbinghaus & Peckhaus, 2015).

In (Bao et al., 2004) the authors described Zermelo's navigation problem on Riemannian manifolds and showed that the path with shortest travel time is the geodesic of Randers metrics. Conversely, they showed constructively that every Randers metric arises as a solution to Zermelo's navigational problem on some Riemannian landscape under the influence of an appropriate wind. See Proposition 1.1 in Section 1.3 in (Bao et al., 2004). The Funk metric on the unit n-dimensional ball $\mathbb{B}^n(1)$ given in Definition 2, which is one of the most important Randers metrics, can be obtained by perturbing the Euclidean metric $\|\cdot\|$ by the vector field $W_x = -x$. A physical interpretation of this could be a lake in the shape of the unit disk with a concentric and symmetric wind current given by the vector field $W(x_1, x_2) = (-x_1, -x_2)$. The distance function (or the shortest traveling time from P to Q) induced by the Funk metric, for $P \neq Q$, is given by:

$$d_F(P,Q) = \ln\left(\frac{\sqrt{\langle P, Q-P\rangle^2 + (1-\|P\|^2)\|Q-P\|^2} - \langle P, Q-P\rangle}{\sqrt{\langle P, Q-P\rangle^2 + (1-\|P\|^2)\|Q-P\|^2} - \langle Q, Q-P\rangle}\right), \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the usual inner product and the usual Euclidean norm, respectively, and $d_F(P,P) = 0$. Both d_F in equation (1.1) and F in Definition 2 could be called Funk metric (see Sadeghi (2021); Shen (2001)). Chávez et al. (2021) proved that d_F given by (1.1) is not reversible ($d_F(P,Q) \neq d_F(Q,P)$) and it is not translation invariant, but the one is rotational invariant around the origin. Additionally, considering $O = (0,0)$, it can be proved that (see Chávez et al. (2021) and Shen (2001))

$$\lim_{\|Q\|\to 1} d_F(O,Q) = +\infty \quad \text{and} \quad \lim_{\|P\|\to 1} d_F(P,O) = \ln 2.$$

In other words, if a boat starts from the origin of the disk towards the boundary, it will take an infinite time to reach the destination, meaning it never reaches the boundary. And if a boat starts from the boundary of $\mathbb{B}^2$ towards the origin of the disk, the minimum travel time is $\ln 2$ units of time. Unlike classical definitions in the literature, such as the one found in Example 1.1.2 in (Shen, 2001) or in (Sadeghi, 2021), the approach adopted here is directly motivated by the navigation problem and aims to simplify the derivation of explicit formulas for distances. While the definition may appear distinct, it is equivalent under appropriate transformations, as discussed in detail in Example 1.2.4 in (Shen, 2001).

More about Funk metrics defined on any strongly convex bounded domain in $\mathbb{R}^n$ can be found in (Funk, 1929; Sadeghi, 2021; Shen, 2001).

In this work, we consider the wind current given by the vector field $W_\lambda(x_1, x_2) = \lambda(-x_1, -x_2)$, where $\lambda \geq 0$. (For $\lambda < 0$ the study is analogous). With this, we define the $\lambda-$Funk metric (see Definition 4 below). Naturally, when $\lambda = 0,$ we obtain the Euclidean norm; when $\lambda = 1$, we get the Funk metric on $\mathbb{B}^2$. We note the $\lambda-$Funk metric is spherically symmetric Finsler metric. While Randers metrics and Zermelo's problem have been extensively studied, the development of λ-Funk metrics in this specific context is original, filling a gap in the literature by generalizing known metrics in physical navigation problems. The λ-Funk metric describes the distance measured in a space where travel cost depends on the direction and intensity of the external flow. This model combines geometric and physical properties to capture the characteristics of Zermelo's navigation problem in a circular lake under perturbations. This study contributes both to the geometric understanding of Finslerian metrics and to the analysis of navigation problems in anisotropic environments. The formulas obtained provide useful tools for modeling and optimizing trajectories in contexts such as maritime and aerial transport.

In Section 2 we recall some basic results for the well development of the work. In Section 3 we obtain and define the $\lambda-$Funk metric, recalling results about spherically symmetric Finsler metrics we prove that their geodesics are straight lines. In Section 4 we obtain the $\lambda-$Funk distance (or traveling time), and we give some properties such as their non-symmetry. In Section 5 we classify the circumference, and with this, we obtain formulas for the distance from point to line and from line to point.

In conclusion, the contributions of the paper include:

1. Formulas for travel time: We derive the corresponding Finsler metric for the navigation problem and obtain the distance formula (travel time) from one point to another.

2. Geometric properties: We present the equation of the circumference in the λ-metric and formulas for distances between a point and a line, as well as between a line and a point.

This study complements previous research on Zermelo's problem and Randers metrics, as explored by authors such as Bao, Shen, and others. In particular, while earlier works focused on the study of Funk metrics in purely geometric or convex contexts, our approach embeds these metrics into a physical context with an external perturbation.

The λ-Funk metric provides a natural generalization, unifying known metrics into a single model that encompasses well-studied specific cases, such as the Euclidean metric ($\lambda = 0$) and the Funk metric ($\lambda = 1$). This work offers novel contributions by explicitly deriving travel time formulas, point-line distances, and circumference equations within this new class of metrics.

2 PRELIMINARIES

In this section, some definitions and results necessary for the development of our work are introduced. We adopt the definitions given in (Chávez et al., 2021), which can also be found in (do Carmo, 2019), such as inner product, norm, regular curve, arc length (which will be referred to as usual or Euclidean), and vector field.

Although a Finsler metric F is a function defined on the Fiber TM of a differentiable manifold M. In this work, we present a simplified definition of a Finsler metric on an open subset U of $\mathbb{R}^n$. For a more comprehensive treatment of Finsler metrics on general manifolds, we refer to Cheng & Shen (2012); Guo & Mo (2018); Shen (2001).

Definition 1. The function $F : U \times \mathbb{R}^n \to \mathbb{R},$ where $U \subset \mathbb{R}^n$, is called *Finsler metric on* U, if for $x \in U$ and $y \in \mathbb{R}^n$, F satisfies the following properties:

1. $F(x, y)$ is C^∞ for all $x \in U$ and $y \neq 0$;
2. $F(x, y) > 0$, for all $x \in U$ and $y \neq 0$;
3. $F(x, \delta y) = \delta F(x, y)$, where δ is any positive real number;
4. The Hessian matrix of $\frac{1}{2}F^2$, denoted by $[g_{ij}]$,

$$[g_{ij}] = \left[\frac{1}{2}\frac{\partial^2 F^2}{\partial y_i \partial y_j}\right]$$

is positive definite.

For example, Euclidean norm, Riemannian metrics or the Funk metric defined below are Finsler metrics.

Definition 2. Let $x = (x_1, x_2) \in \mathbb{B}^2 = \{x \in \mathbb{R}^2;\ x_1^2 + x_2^2 < 1\}$ and $y = (y_1, y_2) \in \mathbb{R}^2$. The function

$$F(x, y) = \sqrt{a_{11}(x)y_1^2 + 2a_{12}(x)y_1y_2 + a_{22}(x)y_2^2} + b_1(x)y_1 + b_2(x)y_2,$$

where $b_1 = \dfrac{x_1}{1 - x_1^2 - x_2^2}$, $b_2 = \dfrac{x_2}{1 - x_1^2 - x_2^2}$ and $[a_{ij}] = \frac{1}{(1-x_1^2-x_2^2)^2}\begin{pmatrix} 1 - x_2^2 & x_1x_2 \\ x_1x_2 & 1 - x_1^2 \end{pmatrix}$, is called the *Funk metric on the unit disk* $\mathbb{B}^2$.

Remembering that the usual Euclidean norm of the vector y is defined by $\|y\| = \sqrt{y_1^2 + y_2^2}$, we have that the function F in the above definition can be interpreted as a generalization or perturbation of the usual Euclidean norm. That is, $F(x, y)$ can be thought of as the “norm” of the vector $y \in \mathbb{R}^2$ at the point $x \in \mathbb{B}^2$. In the language of Finsler geometry, this perturbed “norm” is called the Funk metric on the unit disk $\mathbb{B}^2$. In general, the Finsler distance induced by Finsler metrics are not distance metrics (in the classic sence), because Finsler distance can not be symmetric, since, $F(-v) \neq F(v)$.

Definition 3 (Finsler Arc Length)**.** Let $c : [a, b] \to \mathbb{B}^2 \subset \mathbb{R}^2$ be a piecewise regular curve. The *(Finsler-type) arc length of* c is defined by

$$\mathscr{L}_F(c) := \int_a^b F(c(t), c'(t))dt,$$

where F is a Finsler metric. For any points $P, Q \in \mathbb{B}^2$, the *distance* from P to Q induced by F is defined as

$$d_F(P, Q) := \inf_c \mathscr{L}_F(c),$$

where the infimum is taken over the set of all piecewise regular curves $c : [a, b] \to \mathbb{B}^2$ such that $c(a) = P$ and $c(b) = Q$.

The Finsler-type arc length, when F is a Randers metric can be interpreted as traveling time of a boat sailing along the curve c, from $c(a)$ to $c(b)$, for example, the Funk metric models a boat sailing with unit speed in $\mathbb{B}^2$, where a wind current given by

$W_x = (-x_1, -x_2)$ is present with speed less than 1. (see Section 3 in (Chávez et al., 2021)). The Funk metric in $\mathbb{B}^2$ is a special case of Randers metrics.

A Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is said to be *Projectively flat* if all geodesics are straight lines in U.

That is known that the Funk metric is projectively flat, it is, their *shortest paths* are straight lines (see Example 9.2.1 in (Shen, 2001)). More details on geodesics and their relation to shortest paths can be found in Section 3.2 in (Chern & Shen, 2005) and Section 2.3 in (Cheng & Shen, 2012).

Equation (1.1) can be more manageable using the following version of Theorem 5.1 in (Chávez et al., 2021).

Theorem 1 (Theorem 2.3 in (Chávez et al., 2024))**.** Let P, Q be points in $\mathbb{B}^2$ and $r \geq 1$ be a real number, then

$$d_F(P, Q) = \ln r \iff \left\| \frac{P}{r} - Q \right\| = \frac{r-1}{r},$$

where $\| \cdot \|$ denotes the usual Euclidean norm.

3 ZERMELO NAVIGATION PROBLEM

It is worth noting that any Zermelo navigation problem in $\Omega \subset \mathbb{R}^2$ (including in broader domains like differentiable manifolds) results in a Randers metric on Ω (or on larger domains). Furthermore, every Randers metric originates from a navigation problem (see (Bao et al., 2004; Shen, 2003) and Chapter 2 in (Cheng & Shen, 2012)).

In this section, we will explore Randers metrics derived from the Zermelo navigation problem modeled on an open subset Ω of $\mathbb{R}^2$.

Suppose that a boat is pushed by an internal force (like a motor force) with the velocity vector U_x of constant length, $\|U_x\| = 1$. Without an external velocity vector, the shortest paths are straight lines. In this case, the length of a straight line segment corresponds exactly to the travel time of the boat. Specifically, if $c : [0, t_0] \to \mathbb{R}^2$ is the position vector of the boat, such that $c'(t) = U_{c(t)}$ (a unit velocity vector), then:

$$\int_0^t \|c'(\tau)\| d\tau = \int_0^t 1 d\tau = t.$$

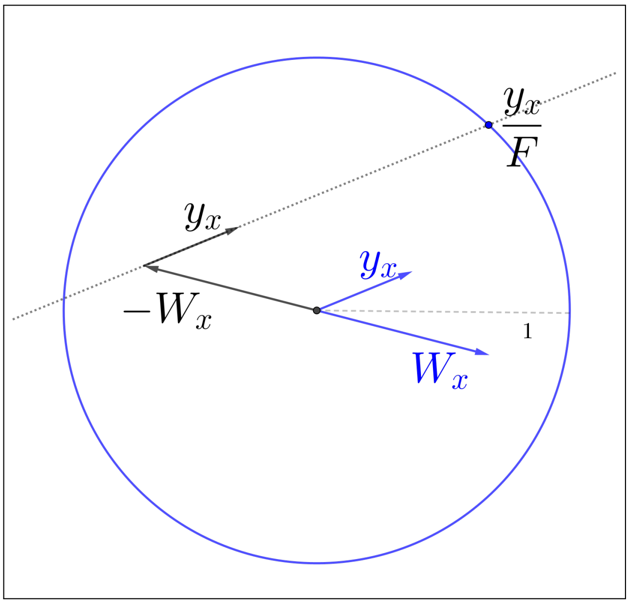
Now, suppose that there exists an external velocity vector W_x, like generated by the wind, with $\|W_x\| < 1$. This condition ensures that the boat can move in all directions. The combination of the two velocity vectors on the object, $T_x = U_x + W_x$, gives the direction and speed of the object at the point $x \in \Omega$. Once the internal velocity vector U_x with $\|U_x\| = 1$ is chosen, we have:

$$\|T_x - W_x\| = \|U_x\| = 1. \tag{3.1}$$

For any vector $y_x \in \mathbb{R}^2$, there exists (see Figure 1) a unique solution $F = F(x, y_x) > 0$ to the following equation:

$$\left\| \frac{y_x}{F(x, y_x)} - W_x \right\| = 1. \tag{3.2}$$

Figure 1 – Existence of F



Source: the authors (2024)

By comparing (3.1) and (3.2), we obtain:

$$F(x, T_x) = 1.$$

If $c : [0, t_0] \to \Omega$ is a smooth curve with $c'(t) = T_{c(t)}$, then the arc length $\mathcal{L}_F(c)$ of c is equals to the travel time of the object along c. Indeed:

$$\mathcal{L}_F(c) = \int_0^{t_0} F(c(\tau), T_{c(\tau)})d\tau = \int_0^{t_0} 1d\tau = t_0.$$

Therefore, in the presence of an external force W_x, the search for shortest paths is no longer in the Euclidean metric, but in the metric F.

We will now derive an expression for the function $F = F(x, y_x)$. To simplify notation, we will omit the subscript x from y_x and W_x going forward. Squaring both sides of Equation (3.2) and expanding the squared norm, we get:

$$\frac{\|y\|^2}{F^2} - 2\frac{\langle y, W\rangle}{F} + \|W\|^2 = 1.$$

Multiplying both sides of the above equality by F^2, we have:

$$(1 - \|W\|^2)F^2 + 2\langle y, W\rangle F - \|y\|^2 = 0.$$

This last equality is a quadratic equation, whose roots are given by:

$$F = -\frac{\langle W, y\rangle}{1 - \|W\|^2} \pm \frac{\sqrt{\langle W, y\rangle^2 + \|y\|^2(1 - \|W\|^2)}}{1 - \|W\|^2}.$$

Now, since:

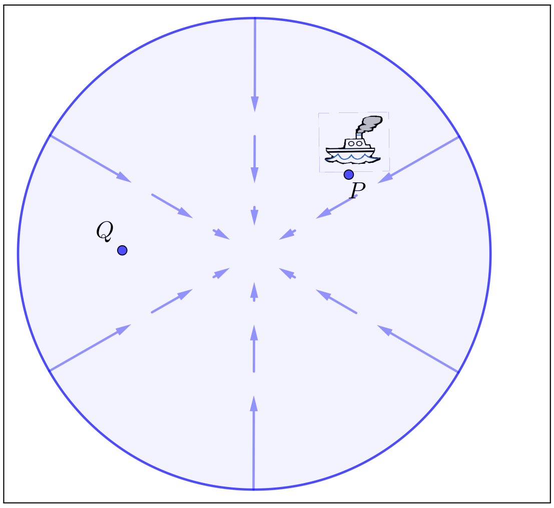
$$\sqrt{\langle W, y\rangle^2 + \|y\|^2(1 - \|W\|^2)} \geq |\langle W, y\rangle| \geq \langle W, y\rangle.$$

Equality holds if and only if $y = 0$. Thus, there will always be one positive root and one non-positive root, ensuring that $F > 0$ for all $y \neq 0$. Thus, we obtain:

$$F = \frac{\sqrt{\langle W, y\rangle^2 + \|y\|^2(1 - \|W\|^2)}}{1 - \|W\|^2} - \frac{\langle W, y\rangle}{1 - \|W\|^2}. \tag{3.3}$$

Given a non-negative constant λ, we are going to define the $\lambda-$Funk metric considering $W = W_x = -\lambda(x_1, x_2)$ (see Figure 2).

Figure 2 – Wind Current: $W_x = -\lambda x$



Source: the authors (2024)

Definition 4. The $\lambda-$*Funk metric* $F : \Omega_\lambda \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$F = \frac{\sqrt{\lambda^2 \langle x, y\rangle^2 + \|y\|^2(1-\lambda^2\|x\|^2)}}{1-\lambda^2\|x\|^2} + \frac{\lambda\langle x, y\rangle}{1-\lambda^2\|x\|^2}, \tag{3.4}$$

where $\Omega_\lambda = \{x \in \mathbb{R}^2; \|x\| < 1/\lambda\}$ for $\lambda > 0$, and $\Omega_0 = \mathbb{R}^n$.

Remark 1. It is straightforward to prove that the function (3.4) satisfies all the properties of a Finsler metric in Definition 1. And, note that F in (3.4) remains the classical Euclidean norm when $\lambda = 0$ and the Funk metric on the unit disk in Definition 2 when $\lambda = 1$.

We recall the following result, which characterizes spherically symmetric Finsler metrics that are projectively flat (i.e. geodesics are straight lines).

Theorem 2 (Theorem 1.1 in (Huang & Mo, 2013))**.** Let $F = \|y\|\phi\left(\|x\|, \frac{\langle x,y\rangle}{\|y\|}\right)$ be a spherically symmetric Finsler metric in $\mathbb{B}^n(\mu) = \{x \in \mathbb{R}^n; \ \|x\| < \mu\}$. Then, F is projectively flat if, and only if, $\phi = \phi(r,s)$ satisfies $r\phi_{ss} - \phi_r + s\phi_{rs} = 0,$ where the sub-index r, s represent the partial derivatives with respect to r and s, respectively.

Theorem 3. The $\lambda-$Funk metric defined in (3.4) is projectively flat.

Proof. Note that (3.4) can be rewritten as $F = \|y\|\phi\left(\|x\|, \frac{\langle x,y\rangle}{\|y\|}\right)$ where

$$\phi(r,s) = \frac{\sqrt{1+\lambda^2(s^2-r^2)}}{1-\lambda^2 r^2} + \frac{\lambda s}{1-\lambda^2 r^2}. \tag{3.5}$$

By differentiating ϕ with respect to r and s in (3.5), we obtain

$$\phi_r = \frac{\lambda^2 r(1+\lambda^2[2s^2-r^2])}{(1-\lambda^2r^2)^2(1+\lambda^2[s^2-r^2])^{1/2}} + \frac{2\lambda^3 sr}{(1-\lambda^2r^2)^2} \tag{3.6}$$

and

$$\phi_s = \frac{\lambda^2 s}{(1-\lambda^2r^2)(1+\lambda^2[s^2-r^2])^{1/2}} + \frac{\lambda}{1-\lambda^2r^2}. \tag{3.7}$$

Now, by differentiating (3.6) and (3.7) with respect to s, we have

$$\phi_{rs} = \frac{\lambda^4 rs(3+\lambda^2[2s^2-3r^2])}{(1-\lambda^2r^2)^2(1+\lambda^2[s^2-r^2])^{3/2}} + \frac{2\lambda^3 r}{(1-\lambda^2r^2)^2} \tag{3.8}$$

and

$$\phi_{ss} = \frac{\lambda^2}{(1+\lambda^2[s^2-r^2])^{3/2}}. \tag{3.9}$$

Thus, using (3.6), (3.8), and (3.9), we obtain

$$\begin{aligned} r\phi_{ss} - \phi_r + s\phi_{rs} &= \frac{\lambda^2 r}{(1+\lambda^2[s^2-r^2])^{3/2}} - \frac{\lambda^2 r(1+\lambda^2[2s^2-r^2])}{(1-\lambda^2r^2)^2(1+\lambda^2[s^2-r^2])^{1/2}} - \frac{2\lambda^3 sr}{(1-\lambda^2r^2)^2} \\ &\quad + \frac{\lambda^4 rs^2(3+\lambda^2[2s^2-3r^2])}{(1-\lambda^2r^2)^2(1+\lambda^2[s^2-r^2])^{3/2}} + \frac{2\lambda^3 rs}{(1-\lambda^2r^2)^2} \\ &= \frac{\lambda^2 r\left\{(1-\lambda^2r^2)^2 - (1+\lambda^2[2s^2-r^2])(1+\lambda^2[s^2-r^2]) + \lambda^2 s^2(3+\lambda^2[2s^2-3r^2])\right\}}{(1-\lambda^2r^2)^2(1+\lambda^2[s^2-r^2])^{3/2}} \\ &= \frac{\lambda^2 r\left\{1+\lambda^2[3s^2-2r^2] + \lambda^4[2s^4-3r^2s^2+r^4] - (1+\lambda^2[2s^2-r^2])(1+\lambda^2[s^2-r^2])\right\}}{(1-\lambda^2r^2)^2(1+\lambda^2[s^2-r^2])^{3/2}} \\ &= 0. \end{aligned}$$

Therefore, by Theorem 2, we have that $\lambda-$Funk metric given by (3.4) is projectively flat on Ω_λ. □

4 INDUCED DISTANCE

By Theorem 3 and inspired by Chávez et al. (2021) we obtain the distance from P to Q denoted by $d_F(P,Q)$.

Theorem 4. Let $\lambda > 0$, $\Omega_\lambda = \mathbb{B}^2(1/\lambda)$. Given $P, Q \in \Omega_\lambda$, then, the distance (or traveling time) induced by the $\lambda-$Funk metric, for $P \neq Q$, is given by:

$$d_F(P,Q) = \frac{1}{\lambda} \ln \left(\frac{\sqrt{\lambda^2 \langle P, Q-P\rangle^2 + (1-\lambda^2\|P\|^2)\|Q-P\|^2} - \lambda\langle P, Q-P\rangle}{\sqrt{\lambda^2 \langle P, Q-P\rangle^2 + (1-\lambda^2\|P\|^2)\|Q-P\|^2} - \lambda\langle Q, Q-P\rangle} \right), \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the usual inner product and the usual Euclidean norm, respectively, and $d_F(P,P) = 0$.

Proof. Let $P, Q \in \Omega_\lambda \subset \mathbb{R}^2$ be distinct points. By Theorem 3, since "$\lambda-$Funk" metric F is projectively flat, we consider the parametric curve $c : [0,1] \to \Omega_\lambda$ defined by:

$$c(t) \;\; = tQ + (1-t)P \tag{4.2}$$

which connects the points P and Q. Note that $c(0) = P$ and $c(1) = Q$. Differentiating (4.2) with respect to t:

$$c'(t) = Q - P \tag{4.3}$$

By the Definition 3, the "$\lambda-$Funk" distance from P to Q, denoted by $d_F(P,Q)$, is given by:

$$d_F(P,Q) = \mathscr{L}_F(c) = \int_0^1 F(c(t), c'(t))\, dt \tag{4.4}$$

where F is given by (3.4). Substituting (4.2) and (4.3) into (4.4), we have:

$$d_F(P,Q) = d_F(P,Q)_1 + d_F(P,Q)_2$$

where

$$d_F(P,Q)_1 = \int_0^1 \frac{\sqrt{\|Q-P\|^2(1-\lambda^2\|P\|^2) + \lambda^2\langle P, Q-P\rangle^2}}{(1-\lambda^2\|P\|^2) - 2\lambda^2\langle P, Q-P\rangle t - \lambda^2\|Q-P\|^2 t^2}\, dt \tag{4.5}$$

and

$$d_F(P,Q)_2 = \int_0^1 \frac{\lambda\|Q-P\|^2 t + \lambda\langle P, Q-P\rangle}{(1-\lambda^2\|P\|^2) - 2\lambda^2\langle P, Q-P\rangle t - \lambda^2\|Q-P\|^2 t^2}\, dt. \tag{4.6}$$

We define k as follows to simplify the notation:

$$k = \|Q-P\|^2(1-\lambda^2\|P\|^2) + \lambda^2\langle P, Q-P\rangle^2. \tag{4.7}$$

Note:

$$(1-\lambda^2\|P\|^2)-2\lambda^2\langle P,Q-P\rangle t-\lambda^2\|Q-P\|^2t^2 = -\lambda^2\|Q-P\|^2\left[t^2+2\frac{\langle P,Q-P\rangle}{\|Q-P\|^2}t-\frac{1-\lambda^2\|P\|^2}{\lambda^2\|Q-P\|^2}\right].$$

Now, factorizing the right-hand side of the above equation, we obtain:

$$(1-\lambda^2\|P\|^2)-2\lambda^2\langle P,Q-P\rangle t-\lambda^2\|Q-P\|^2t^2 = -\lambda^2\|Q-P\|^2(t-\tau_1)(t-\tau_2) \tag{4.8}$$

where

$$\tau_1 = \frac{-\lambda\langle P,Q-P\rangle-\sqrt{k}}{\lambda\|Q-P\|^2} \tag{4.9}$$

and

$$\tau_2 = \frac{-\lambda\langle P,Q-P\rangle+\sqrt{k}}{\lambda\|Q-P\|^2}. \tag{4.10}$$

Substituting (4.7) and (4.8) into (4.5), we have:

$$d_F(P,Q)_1 = -\frac{\sqrt{k}}{\lambda^2\|Q-P\|^2}\int_0^1 \frac{1}{(t-\tau_1)(t-\tau_2)}\,dt.$$

This time, using the method of partial fractions, we have:

$$d_F(P,Q)_1 = -\frac{\sqrt{k}}{\lambda^2\|Q-P\|^2}\int_0^1 \left[\frac{1}{(\tau_1-\tau_2)(t-\tau_1)}-\frac{1}{(\tau_1-\tau_2)(t-\tau_2)}\right]dt.$$

Therefore, integrating, we obtain:

$$\begin{aligned} d(P,Q)_1 &= -\frac{\sqrt{k}}{\lambda^2\|Q-P\|^2}\left[\frac{\ln|t-\tau_1|}{\tau_1-\tau_2} - \frac{\ln|t-\tau_2|}{\tau_1-\tau_2}\right]_0^1 \\ &= -\frac{\sqrt{k}}{\lambda^2\|Q-P\|^2}\frac{1}{(\tau_1-\tau_2)}\left[\ln\left|\frac{t-\tau_1}{t-\tau_2}\right|\right]_0^1 \\ &= -\frac{\sqrt{k}}{\lambda^2\|Q-P\|^2}\frac{1}{(\tau_1-\tau_2)}\ln\left|\frac{\tau_2(1-\tau_1)}{\tau_1(1-\tau_2)}\right|. \end{aligned} \tag{4.11}$$

Note that, by (4.9) and (4.10), we have:

$$\frac{1}{\tau_1-\tau_2} = -\frac{\lambda\|Q-P\|^2}{2\sqrt{k}}. \tag{4.12}$$

Thus, substituting (4.12) into (4.11), we have:

$$d_F(P,Q)_1 = \frac{1}{2\lambda}\ln\left|\frac{\tau_2(1-\tau_1)}{\tau_1(1-\tau_2)}\right|. \tag{4.13}$$

On the other hand, note

$$[(1-\lambda^2\|P\|^2) - 2\lambda^2\langle P, Q-P\rangle t - \lambda^2\|Q-P\|^2t^2]' = -2\lambda^2[\|Q-P\|^2 t + \langle P, Q-P\rangle]. \tag{4.14}$$

Thus, substituting (4.14) into (4.6), we have:

$$d_F(P,Q)_2 = -\frac{1}{2\lambda}\int_0^1 \frac{[(1-\lambda^2\|P\|^2) - 2\lambda^2\langle P, Q-P\rangle t - \lambda^2\|Q-P\|^2t^2]'}{(1-\lambda^2\|P\|^2) - 2\lambda^2\langle P, Q-P\rangle t - \lambda^2\|Q-P\|^2t^2}\,dt.$$

By the Fundamental Theorem of Calculus, we obtain:

$$d_F(P,Q)_2 = -\frac{1}{2\lambda}\left[\ln\left|(1-\lambda^2\|P\|^2) - 2\lambda^2\langle P, Q-P\rangle t - \lambda^2\|Q-P\|^2t^2\right|\right]_0^1. \tag{4.15}$$

Thus, substituting (4.8) into (4.15), we have:

$$d_F(P,Q)_2 = -\frac{1}{2\lambda}\left[\ln\left|-\lambda^2\|Q-P\|^2(t-\tau_1)(t-\tau_2)\right|\right]_0^1$$

or, equivalently, we obtain:

$$d_F(P,Q)_2 = -\frac{1}{2\lambda}\ln\left|\frac{(1-\tau_1)(1-\tau_2)}{\tau_1\tau_2}\right|. \tag{4.16}$$

Substituting (4.12) and (4.16) into (4.4), we have:

$$\begin{aligned} d_F(P,Q) &= \frac{1}{2\lambda}\ln\left|\frac{\tau_2(1-\tau_1)}{\tau_1(1-\tau_2)}\right| - \frac{1}{2\lambda}\ln\left|\frac{(1-\tau_1)(1-\tau_2)}{\tau_1\tau_2}\right| \\ &= \frac{1}{2\lambda}\ln\left(\frac{\tau_2^2}{(1-\tau_2)^2}\right) \\ &= \frac{1}{\lambda}\ln\left|\frac{\tau_2}{\tau_2-1}\right|. \end{aligned} \tag{4.17}$$

Claim 1. $\tau_2 > 1$.

In fact, note that, from the definition of τ_2 in (4.10), we have $\tau_2 > 1$ if and only if

$$\sqrt{\lambda^2\langle P, Q-P\rangle^2 + (1-\lambda^2\|P\|^2)\|Q-P\|^2} > \lambda(\langle P, Q-P\rangle + \|Q-P\|^2). \tag{4.18}$$

Now, two situations arise:

- If $\langle P, Q-P\rangle + \|Q-P\|^2 < 0$, then (4.18) is clearly true.
- If $\langle P, Q-P\rangle + \|Q-P\|^2 \geq 0$, then (4.18) is true if and only if

$$\begin{aligned} &\lambda^2\langle P, Q-P\rangle^2 + (1-\lambda^2\|P\|^2)\|Q-P\|^2 > \lambda^2\left(\langle P, Q-P\rangle^2 + 2\langle P, Q-P\rangle\|Q-P\|^2 + \|Q-P\|^4\right) \\ &\Leftrightarrow (1-\lambda^2\|P\|^2)\|Q-P\|^2 > 2\lambda^2\langle P, Q-P\rangle\|Q-P\|^2 + \lambda^2\|Q-P\|^4 \end{aligned}$$

Since $P \neq Q$, dividing both sides of the above inequality by $\|Q-P\|^2$, we have:

$$\begin{aligned} 1-\lambda^2\|P\|^2 > 2\lambda^2\langle P, Q-P\rangle + \lambda^2\|Q-P\|^2 \quad &\Leftrightarrow \quad 1 > \lambda^2\left(\|P\|^2 + 2\langle P, Q-P\rangle + \|Q-P\|^2\right) \\ &\Leftrightarrow \quad 1 > \lambda^2\|P + (Q-P)\|^2 \\ &\Leftrightarrow \quad \frac{1}{\lambda^2} > \|Q\|^2, \end{aligned}$$

which is also true.

Therefore, in either case, (4.18) is true. This concludes the proof of the claim. Using Claim 1 in (4.17), we have that the distance from P to Q is given by:

$$d_F(P,Q) = \frac{1}{\lambda}\ln\left(\frac{\tau_2}{\tau_2 - 1}\right). \tag{4.19}$$

Replacing (4.10) in (4.19), and using

$$\langle P, Q-P\rangle + \|Q-P\|^2 = \langle Q, Q-P\rangle$$

we obtain the result. □

Remark 2. For $\lambda \neq 0$, we have

1. d_F is not symmetric. In fact, consider O as the origin and P any point in Ω_λ distinct from O. Thus, from equation (4.1) in Theorem 4, we observe:

$$\begin{aligned} d_F(O,P) &= \frac{1}{\lambda}\ln\left(\frac{\sqrt{\|P\|^2}}{\sqrt{\|P\|^2 - \lambda\langle P,P\rangle}}\right) \\ &= \frac{1}{\lambda}\ln\left(\frac{1}{1-\lambda\|P\|}\right) \\ &= -\frac{1}{\lambda}\ln(1-\lambda\|P\|) \\ &\neq \frac{1}{\lambda}\ln(1+\lambda\|P\|) \\ &= \frac{1}{\lambda}\ln\left(\frac{\sqrt{\lambda^2\|P\|^4 + (1-\lambda^2\|P\|^2)\|P\|^2} - \lambda\langle P,-P\rangle}{\sqrt{\lambda^2\|P\|^4 + (1-\lambda^2\|P\|^2)\|P\|^2}}\right) = d_F(P,O). \end{aligned}$$

   Theorem 5, below, provides a better visualization of this asymmetry.

2. d_F is not invariant under translations. In fact, consider $T : \Omega_\lambda \to \Omega_\lambda$ given by $T(x) = x + P_0$ where $P_0 \in \Omega_\lambda \setminus \{O\}$. Note:

$$d_F(O,-P_0) = d_F(O,P_0) \neq d_F(P_0,O) = d_F(T(O),T(-P_0)).$$

3. d_F is invariant under rotations around the origin. It suffices to note that the Euclidean inner product and the Euclidean norm are invariant under rotations around the origin.

5 GEOMETRY OF THE $\lambda-$FUNK METRIC ON Ω_λ

In this section, we will examine some geometric properties of basic geometry, such as the distance from one point to another, the distance from a point to a line, and the study of the circle, using the distance obtained in (4.1) induced by the $\lambda-$Funk metric (3.4).

Theorem 5. Let $P,\ Q$ be points in Ω_λ ($\lambda > 0$) and $r \geq 1$ a real number, then

$$d_F(P,Q) = \frac{1}{\lambda}\ln r \iff \left\|\frac{P}{r} - Q\right\| = \frac{1}{\lambda}\left(\frac{r-1}{r}\right). \tag{5.1}$$

Proof. If $r = 1$, we have

$$d_F(P,Q) = \frac{1}{\lambda}\ln 1 = 0 \Leftrightarrow P = Q \Leftrightarrow \frac{1}{\lambda}\left(\frac{1-1}{1}\right) = 0 = \frac{1}{\lambda}\|P - Q\| = \frac{1}{\lambda}\left\|\frac{P}{1} - Q\right\|.$$

If $r > 1$, we have

$$\left\|\frac{P}{r} - Q\right\| = \frac{1}{\lambda}\left(\frac{r-1}{r}\right) \Leftrightarrow \left\|P\left(\frac{r-1}{r}\right) + (Q-P)\right\| = \frac{1}{\lambda}\frac{r-1}{r} \Leftrightarrow \left\|P + \left(\frac{r}{r-1}\right)(Q-P)\right\|^2 = \frac{1}{\lambda^2}$$

$$\Leftrightarrow 1 - \lambda^2\|P\|^2 = \lambda^2\left(\frac{r}{r-1}\right)^2\|Q-P\|^2 + 2\lambda^2\left(\frac{r}{r-1}\right)\langle P, Q-P\rangle.$$

Note that $P \neq Q$ since $r > 1$, thus $\|P - Q\| \neq 0$ and $r - 1 \neq 0$. Therefore, multiplying the above equality by $(r-1)^2\|Q-P\|^2$ we get

$$\Leftrightarrow (r-1)^2(1-\lambda^2\|P\|^2)\|Q-P\|^2 = \lambda^2 r^2\|Q-P\|^4 + 2\lambda^2 r(r-1)\langle P, Q-P\rangle\|Q-P\|^2$$

$$\Leftrightarrow \begin{aligned} &(r-1)^2\left(\lambda^2\langle P, Q-P\rangle^2 + \|Q-P\|^2(1-\lambda^2\|P\|^2)\right) \\ &\quad = \lambda^2\left((r-1)\langle P, Q-P\rangle + r\|Q-P\|^2\right)^2. \end{aligned} \tag{5.2}$$

Now, since $\langle Q, rQ-P\rangle < \dfrac{1}{\lambda}\|rQ-P\| = \dfrac{r-1}{\lambda^2}$ (since $Q \in \Omega_\lambda$) and $r - 1 > 0,$ we have

$$\begin{aligned} \lambda^2\left[(r-1)\langle P, Q-P\rangle + r\|Q-P\|^2\right] &= \lambda^2\langle rQ-P, Q-P\rangle \\ &= \lambda^2\left[\|rQ-P\|^2 - (r-1)\langle Q, rQ-P\rangle\right] \\ &> \lambda^2[(r-1)^2 - (r-1)^2] = 0. \end{aligned}$$

Thus, being $r-1>0$, (5.2) is equivalent to

$$\Leftrightarrow (r-1)\sqrt{\lambda^2\langle P,Q-P\rangle^2+\|Q-P\|^2(1-\lambda^2\|P\|^2)} = \lambda[(r-1)\langle P,Q-P\rangle + r\|Q-P\|^2]. \quad (5.3)$$

Since $Q\in\Omega_\lambda$ and $P\neq Q$ we have that

$$\lambda^2[\langle Q,Q-P\rangle^2-\langle P,Q-P\rangle^2] = \lambda^2\left[\|Q\|^2-\|P\|^2\right]\|Q-P\|^2 \neq (1-\lambda^2\|P\|^2)\|Q-P\|^2$$

$$\Leftrightarrow \sqrt{\lambda^2\langle P,Q-P\rangle^2+(1-\lambda^2\|P\|^2)\|Q-P\|^2}-\lambda\langle Q,Q-P\rangle \neq 0.$$

Therefore, we obtain that (5.3) is equivalent to

$$\Leftrightarrow r = \frac{\sqrt{\lambda^2\langle P,Q-P\rangle^2+(1-\lambda^2\|P\|^2)\|Q-P\|^2}-\lambda\langle P,Q-P\rangle}{\sqrt{\lambda^2\langle P,Q-P\rangle^2+(1-\lambda^2\|P\|^2)\|Q-P\|^2}-\lambda\langle Q,Q-P\rangle}$$

$$\Leftrightarrow \frac{1}{\lambda}\ln r = \frac{1}{\lambda}\ln\left(\frac{\sqrt{\lambda^2\langle P,Q-P\rangle^2+(1-\lambda^2\|P\|^2)\|Q-P\|^2}-\lambda\langle P,Q-P\rangle}{\sqrt{\lambda^2\langle P,Q-P\rangle^2+(1-\lambda^2\|P\|^2)\|Q-P\|^2}-\lambda\langle Q,Q-P\rangle}\right) = d_F(P,Q).$$

□

Note that, in the proof of the previous theorem, the properties of the Euclidean norm were used, that is, the theorem remains valid for points P and Q in $\mathbb{B}^n(1/\lambda)$, $n\geq 2$.

Remark 3. Note that using equation (5.1) it is easier to show that d_F is invariant under rotations around the origin, that is,

$$d_F(RP,RQ) = d_F(P,Q), \quad (5.4)$$

where $P,Q\in\Omega_\lambda$ and $R=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$.

Remark 4. Consider $P=(0,0)$ in (5.1), then $\|Q\| = \dfrac{1}{\lambda}\dfrac{r-1}{r}$. When $\|Q\|\to\frac{1}{\lambda}$, that is, when Q approaches the boundary of Ω_λ then $r\to+\infty$, consequently $d_F(P,Q)\to+\infty$. This means, together with Property 2 in Remark 2.8 in Chávez et al. (2021), that from any point in Ω_λ, our boat will not be able to leave Ω_λ. On the other hand, consider $Q=(0,0)$ in (5.1), then $\|P\| = \dfrac{r-1}{\lambda}$. When $\|P\|\to\frac{1}{\lambda}$, then $r\to 2$, consequently $d_F(P,Q)\to\frac{\ln 2}{\lambda}$. Thus, from the boundary of Ω_λ, our boat can reach the origin in a time equal to $\dfrac{\ln 2}{\lambda}$.

5.1 Circumference

Since the Funk distance is not symmetric, that is, $d_F(P,Q) \neq d_F(Q,P)$, we have two interpretations for the notion of circumference, these are, the "output" from the center, called *type 1*; and the "input" to the center, called *type 2*.

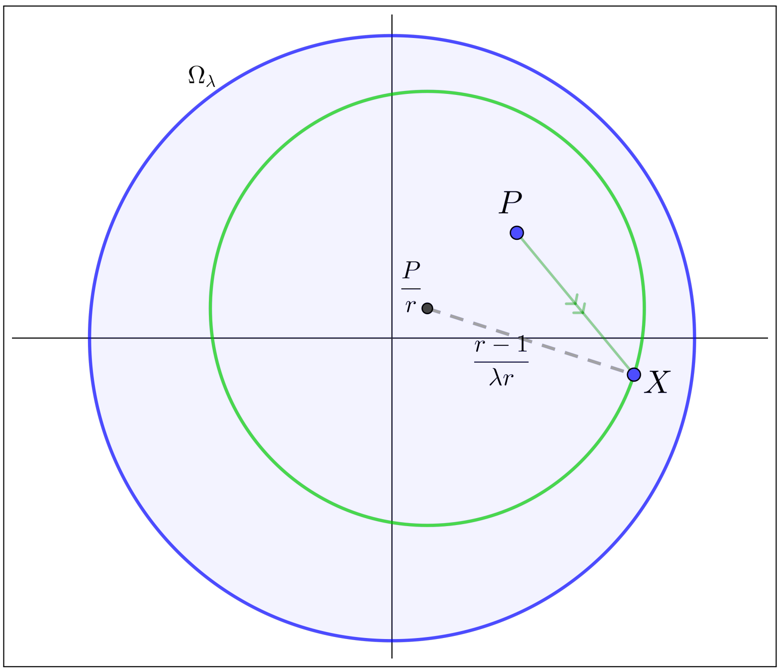
Definition 5. Given P a point in Ω_λ and $r \geq 1$ a real number, we define the *type 1 Funk circumference*, with center P and radius $\frac{\ln r}{\lambda}$, as the points $X \in \Omega_\lambda$ that satisfy the following equation:

$$d_F(P,X) = \frac{\ln r}{\lambda}.$$

By (5.1), we have that the equation of the type 1 Funk circumference with center $P=(a,b)$ and radius $\frac{\ln r}{\lambda}$, is given by:

$$\left(x_1 - \frac{a}{r}\right)^2 + \left(x_2 - \frac{b}{r}\right)^2 = \left(\frac{r-1}{\lambda r}\right)^2. \tag{5.5}$$

Figure 3 – $d_F(P,X) = \frac{\ln r}{\lambda}$



Source: the authors (2024)

Equation (5.5) describes the graph of a Euclidean circle in $\mathbb{R}^2$ with center at $\frac{P}{r}$ and radius $\frac{r-1}{\lambda r}$ (see Figure 3).

Definition 6. Given P a point in Ω_λ and $r \geq 1$ a real number, we define the *type 2 Funk circumference*, with center P and radius $\dfrac{\ln r}{\lambda}$, as the points $X \in \Omega_\lambda$ that satisfy the following equation:

$$d_F(X,P) = \frac{\ln r}{\lambda}.$$

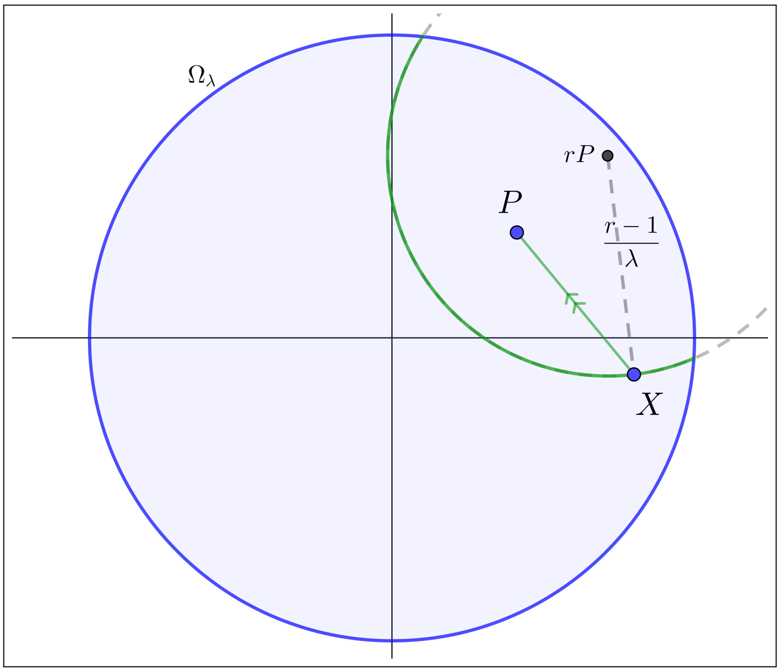
By (5.1), we have that the equation of the type 2 Funk circumference with center $P=(a,b)$ and radius $\dfrac{\ln r}{\lambda}$, is part of the Euclidean circle with center at (ra, rb) and radius $\dfrac{r-1}{\lambda}$. (see Figure 4).

$$\left(a - \frac{x_1}{r}\right)^2 + \left(b - \frac{x_2}{r}\right)^2 = \left(\frac{r-1}{\lambda r}\right)^2,$$

or, equivalently,

$$(x_1 - ra)^2 + (x_2 - rb)^2 = \left(\frac{r-1}{\lambda}\right)^2. \qquad (5.6)$$

Figure 4 – $d_F(X,P) = \frac{\ln r}{\lambda}$



Source: the authors (2024)

5.2 $\lambda-$Funk distance from a Line to a Point

Definition 7. Let s a line in $\mathbb{R}^2$. We define the *distance*, $d_F(s_\lambda, Q)$, from $s_\lambda = s \cap \Omega_\lambda$ to the point $Q \in \Omega_\lambda$ as:

$$d_F(s_\lambda, Q) := \min\{d_F(P', Q);\ P' \in s_\lambda\}.$$

We say that the point $P* \in s_\lambda$ *realizes the distance from s_λ to the point Q*, if

$$d_F(P*, Q) \leq d_F(P', Q), \text{ for all } P' \in s_\lambda.$$

Theorem 6. Let $s : x_2 = mx_1 + c$ a line in $\mathbb{R}^2$. The $\lambda-$Funk distance from $s_\lambda = s \cap \Omega_\lambda$ to the point $Q = (a, b) \in \Omega_\lambda$ is given by:

$$d_F(s_\lambda, Q) = \frac{\ln r}{\lambda} = \frac{1}{\lambda}\ln\left(\frac{1 - \lambda^2 c\cos\theta(b\cos\theta - a\sin\theta) + \lambda|c\cos\theta - b\cos\theta + a\sin\theta|}{1 - \lambda^2(b\cos\theta - a\sin\theta)^2}\right), \tag{5.7}$$

where $\theta = \arctan m.$ And, this distance is realized by

$$P* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} ar\cos\theta + br\sin\theta \\ c\cos\theta \end{pmatrix}. \tag{5.8}$$

Proof. Given a line $s :\ x_2 = mx_1 + c$ and a point $Q = (a, b) \in \Omega_\lambda$, we will first analyze the particular case where $m = 0$ and then analyze the general case. To determine the distance from the line s to the point Q, we take a Funk circumference containing Q and centered at some point on the line. The radius is $\frac{\ln r}{\lambda}$ and the center is at $P = (x_1, c) \in s$. According to Equation (5.6), we have:

$$(x_1 - ra)^2 + (c - rb)^2 = \left(\frac{r-1}{\lambda}\right)^2. \tag{5.9}$$

This is a quadratic equation in the variable x_1, and depending on r, it is possible to find two, one, or no solutions for x_1. We are only interested in finding a single solution, the one that minimizes the distance. Thus, from Equation (5.9), we have the uniqueness conditions, $x_1 = ra$ and

$$(c - rb)^2 = \left(\frac{r-1}{\lambda}\right)^2.$$

Expanding the squares, we get a quadratic equation in r:

$$(1 - \lambda^2 b^2) r^2 + 2(\lambda^2 bc - 1) r + (1 - \lambda^2 c^2) = 0,$$

whose roots are given by:

$$r = \frac{(1 - \lambda^2 bc) \pm \sqrt{(1 - \lambda^2 bc)^2 - (1 - \lambda^2 b^2)(1 - \lambda^2 c^2)}}{1 - \lambda^2 b^2}. \tag{5.10}$$

Note that:

$$(1 - \lambda^2 bc)^2 - (1 - \lambda^2 b^2)(1 - \lambda^2 c^2) = \lambda^2 (c - b)^2.$$

Thus, equation (5.10) becomes $r = \dfrac{1 - \lambda^2 bc \pm \lambda |c - b|}{1 - \lambda^2 b^2}$. Remembering that $r > 1$, we obtain:

$$r = \frac{1 - \lambda^2 bc + \lambda |c - b|}{1 - \lambda^2 b^2}. \tag{5.11}$$

Thus, the Funk distance from s_λ to the point $Q = (a, b)$, for the case where $m = 0$, is given by:

$$d_F(s_\lambda, Q) = \frac{\ln r}{\lambda} = \frac{1}{\lambda} \ln \left(\frac{1 - \lambda^2 bc + \lambda |c - b|}{1 - \lambda^2 b^2} \right). \tag{5.12}$$

This distance is realized by the point, $P* = (ar, c) \in s$. We will later see how to determine this distance for any line s. To do this, we first observe that the force field is symmetric with respect to the origin (see equation (5.4)). Thus, we can apply a rotation around the origin to the axes and find values for x_1 and r based on the results already obtained.

Given a line $s: \ x_2 = mx_1 + c$, we rotate the coordinate axes by θ degrees, such that $m = \tan\theta$. Note that, in the new rotated axes, the line s is a horizontal line (parallel to the x_1 axis), falling into the particular case where $m = 0$. We determine the new coordinates of $Q = (a, b)$:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus, the new coordinates of Q are $(a\cos\theta + b\sin\theta, b\cos\theta - a\sin\theta)$.

From (5.9) and (5.11), we obtain:

$$r = \frac{1 - \lambda^2 c\cos\theta(b\cos\theta - a\sin\theta) + \lambda|c\cos\theta - b\cos\theta + a\sin\theta|}{1 - \lambda^2(b\cos\theta - a\sin\theta)^2} \quad \text{and} \quad x = r(a\cos\theta + b\sin\theta).$$

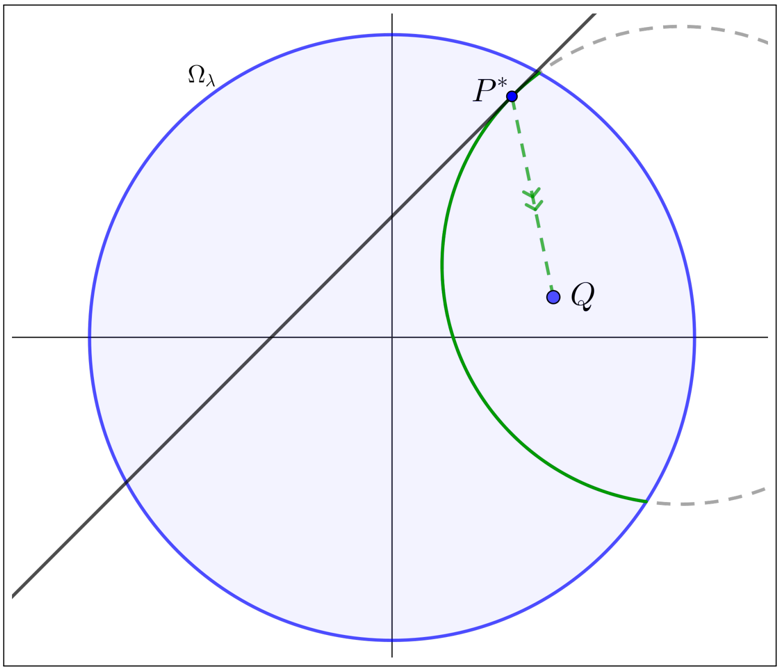
Thus, we obtain the Funk distance from the line s to the point Q, given by (5.7). Moreover, this distance is realized by the point $P** = (r(a\cos\theta + b\sin\theta), c\cos\theta)$. And, to obtain the point on the original line (without rotation around the origin), it is sufficient to rotate the line back to its original position. □

Remark 5. *The distance $d_F(\tilde{s}_\lambda, \tilde{Q})$ where $\tilde{s}_\lambda : x^1 = \tilde{c}$ and $\tilde{Q} = (\tilde{a}, \tilde{b})$, by clockwise rotation of the coordinate axis, is equivalent to obtain the distance $d_F(s_\lambda, Q)$, where $s_\lambda : x^2 = \tilde{c}$ and $P = (-\tilde{b}, \tilde{a})$. That is, from equation* (5.12)*,*

$$d_F(\tilde{s}_\lambda, \tilde{Q}) = d_F(s_\lambda, Q) = \frac{1}{\lambda}\ln\left(\frac{1 - \lambda^2\tilde{a}\tilde{c} + |\tilde{c} - \tilde{a}|}{1 - \lambda^2\tilde{a}}\right).$$

Example 1. Considering the vector field $W(x_1, x_2) = -2/5(x_1, x_2)$, the line $s : x_2 = x_1 + 1$, and the point $Q = (1, 1/10)$, (see Figure 5). From Theorem 6, we have, $d_F(s_\lambda, Q) = 1.36$, and $P^* = (0.45, 1, 45)$.

Figure 5 – $d_F(s_\lambda, Q) = d_F(P^*, Q) = 1.36$



Source: the authors (2024)

5.3 $\lambda-$Funk Distance from a Point to a Line

Definition 8. Let s a line in $\mathbb{R}^2$. We define the *distance*, $d_F(P, s_\lambda)$, from the point $P \in \Omega_\lambda$ to $s_\lambda = s \cap \Omega_\lambda$ as:

$$d_F(P, s_\lambda) := \min\{d_F(P, Q');\ Q' \in s_\lambda\}.$$

We say that the point $Q* \in s_\lambda$ *realizes the distance from* P *to* s_λ, if

$d_F(P, Q*) \leq d_F(P, Q'),$ for all $Q' \in s_\lambda$.

Theorem 7. Let $s\ :\ x_2\ =\ mx_1 + c$ a line in $\mathbb{R}^2$. The $\lambda-$Funk distance from the point $P = (a, b) \in \Omega_\lambda$ to $s_\lambda = s \cap \Omega_\lambda$ is given by:

$$d_F(P, s_\lambda) = \frac{\ln r}{\lambda} = \frac{1}{\lambda} \ln\left(\frac{1 - \lambda^2 c\cos\theta(b\cos\theta - a\sin\theta) + \lambda|c\cos\theta - b\cos\theta + a\sin\theta|}{1 - \lambda^2 c^2 \cos^2\theta}\right), \quad (5.13)$$

where $\theta = \arctan m$. And, this distance is realized by

$$Q* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{a}{r}\cos\theta + \frac{b}{r}\sin\theta \\ c\cos\theta \end{pmatrix}. \quad (5.14)$$

Proof. Given the point $P = (a, b)$ and the line $s: \ x_2 = mx_1 + c$, we will proceed similarly as we did before to determine the Funk distance from the line to the point. We take a Funk circumference centered at P, with radius $\frac{\ln r}{\lambda}$ and intersecting with the line s. We take the point $Q \in s$ to be one of these intersections. First, we will verify the particular case where $m = 0$. Thus, by Equation (5.1), we have:

$$\left(x_1 - \frac{a}{r}\right)^2 + \left(c - \frac{b}{r}\right)^2 = \left(\frac{r-1}{\lambda r}\right)^2. \tag{5.15}$$

Similarly, this is a quadratic equation in the variable x_1, and depending on r. However, we are only interested in finding a single solution, the one that minimizes the distance. To obtain a unique solution of (5.15), the discriminant of the quadratic equation in x_1 above must be equal to zero. Consequently, we will have a quadratic equation in r:

$$r^2(\lambda^2 c^2 - 1) + 2r(1 - bc\lambda^2) + \lambda^2 b^2 - 1 = 0,$$

whose roots are given by:

$$r = \frac{(\lambda^2 bc - 1) \pm \sqrt{(\lambda^2 bc - 1)^2 - (\lambda^2 c^2 - 1)(\lambda^2 b^2 - 1)}}{\lambda^2 c^2 - 1}.$$

We observe:

$$(\lambda^2 bc - 1)^2 - (\lambda^2 c^2 - 1)(\lambda^2 b^2 - 1) = \lambda^2 (c - b)^2.$$

Thus, $r = \frac{(\lambda^2 bc - 1) \pm \lambda |c - b|}{\lambda^2 c^2 - 1}$. To ensure that $r > 1$, we have $r = \frac{\lambda^2 bc - 1 - \lambda |c - b|}{\lambda^2 c^2 - 1}$ and $x_1 = \frac{a}{r}$. Thus, the Funk distance from the point P to the line s (case where $m = 0$) is given by:

$$d_F(P, s) = \frac{\ln r}{\lambda} = \frac{1}{\lambda} \ln \left(\frac{\lambda^2 bc - 1 - \lambda |c - b|}{\lambda^2 c^2 - 1}\right). \tag{5.16}$$

This distance is realized by the point $Q^* = \left(\frac{a}{r}, c\right)$.

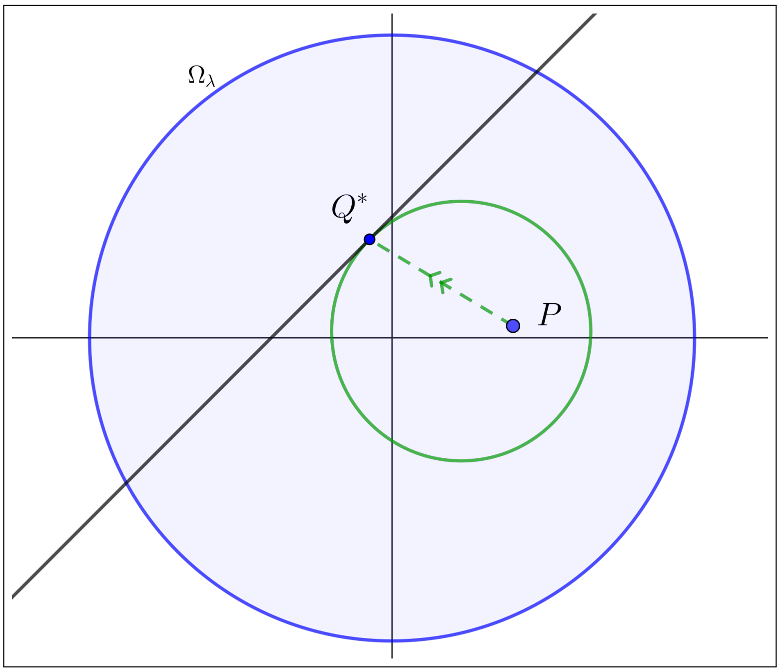
For the case where s is any line, rotate the axes, as we did before, and we obtain (7) which is realized by the point $Q^{**} = \left(\frac{a\cos\theta + b\sin\theta}{r}, c\cos\theta\right)$. Then, we rotate back to obtain the point on the original line. □

Remark 6. *The distance $d_F(\tilde{P},\tilde{s}_\lambda)$ where $\tilde{s}_\lambda : x^1 = \tilde{c}$ and $\tilde{Q} = (\tilde{a},\tilde{b})$, by clockwise rotation of the coordinate axis, is equivalent to obtain the distance $d_F(P,s_\lambda)$, where $s_\lambda : x^2 = \tilde{c}$ and $P = (-\tilde{b},\tilde{a})$. That is, from equation* (5.16)*,*

$$d_F(\tilde{P},\tilde{s}_\lambda) = d_F(P,s_\lambda) = \frac{1}{\lambda}\ln\left(\frac{\lambda^2\tilde{a}\tilde{c} - 1 - \lambda|\tilde{a}-\tilde{c}|}{\lambda^2\tilde{c}^2 - 1}\right).$$

Example 2. Considering the vector field $W(x_1,x_2) = -2/5(x_1,x_2)$, the line $s : x_2 = x_1 + 1$, and the point $P = (1,1/10)$, (see Figure 6). From Theorem 7, we have, $d_F(Q,s_\lambda) = 1.4$, and $Q^* = (-0.19, 0.81)$.

Figure 6 – $d_F(P,s_\lambda) = d_F(P,Q^*) = 1.4$



Source: the authors (2024)

6 CONCLUSION

The Zermelo's navigation problem considered in this work was a boat navigating through a lake-like suitable disk, and the wind current was modeled by vector field $W_\lambda(x_1,x_2) = \lambda(-x_1,-x_2)$, where $\lambda \geq 0$. With this, we defined the $\lambda-$Funk metric (Definition 4) which generalize the usual Euclidean norm ($\lambda = 0$) and the Funk metric given by (1.1) ($\lambda = 1$). We prove that $\lambda-$Funk metric is spherically symmetric Finsler metric, with this, we prove the shortest path are straight lines (Theorem 3). The time traveling or the called $\lambda-$Funk distance induced by the $\lambda-$Funk metric was obtained in Theorem 4. Although the expression of this distance is complicated, Theorem 5 gives us a useful tool to obtain the equation of the circumferences (equations (5.5) and (4.16)), the distance from line to point (Theorem 6) and from point to line (Theorem 7).

The λ-Funk metrics have potential applications in trajectory optimization and route planning in environments subject to external flows, such as ocean currents or winds. Moreover, the geometric study presented here provides analytical tools that can be applied to other areas, including control theory, dynamical systems analysis, and even computational modeling. The explicit formulas derived in this paper lay a foundation for future studies involving more complex geometric or physical problems.

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