UFSM

CIÊNCIA e NATURA

ISSN 2179-460X

OPEN ACCESS

Ci. e Nat., Santa Maria, v. 46, n. spe. 1, e87459, 2024 • https://doi.org/10.5902/2179460X87459
Submitted: 04/11/2024 • Approved: 08/28/2024 • Published: xx/xx/xx

ERMAC e ENMC

A solution to the time-dependent two-dimensional Navier-Stokes equation in a rectangular domain using the Adomian decomposition method and theory of Gröbner Basis

Uma solução para a equação bidimensional de Navier-Stokes dependente do tempo em um domínio retangular utilizando o método de decomposição de Adomian e a teoria de Base de Gröbner

Luciana Rossato Piovesan [I], **Alexandre Sacco de Athayde** [I], **Bardo Ernst Josef Bodmann** [II], **Marco Túllio Menna Barreto de Vilhena** [II],

[I]Universidade Federal de Pelotas, Pelotas, RS, Brasil
[II]Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brasil

ABSTRACT

In the present work we propose a modified decomposition method to derive approximate solutions for non-linear problems. Depending on the type of non-linearity, the source terms of the differential equations to be solved in each recursion step may result in extensive expressions, impractical for computational implementations and applications. This shortcomings are circumvented by the present methodology, which contemplates as a solution procedure in each recursion step a combined variable separation method together with Duhamel's principle, where the non-linearity appears as inhomogeneity. The source terms of the equation in each step of recursion are interpolated by polynomials and, using the Gröbner basis of the set points, the polynomial of reduced degree is obtained so that the integration may be carried out easily. As an application we considered a simplified version of the Navier-Stokes equation, which was used to simulate the wind field making use of the micrometeorological data from the Copenhagen experiment. The derived solution was evaluated against these experimental data from the field experiments showed that the computed results are acceptable and thus the solution may be considered an acceptable one and may be used as a simulation device for these type of field experiments. For almost all experiments twenty eigenvalues and ten recursion steps were sufficient. As results the wind speed at certain positions was simulated and compared to the measured values. The results obtained allow us to affirm that the presented methodology works satisfactorily and, therefore, can be considered a promising tool for solving non-linear problems, which are not tractable with the conventional decomposition method

Keywords: Navier-Stokes equation; Gröbner basis; Two-dimensional interpolation; Polynomial of smallest degree

RESUMO

No presente trabalho propomos um método de decomposição modificado para derivar soluções aproximadas para problemas não lineares. Dependendo do tipo de não linearidade, os termos fontes das equações diferenciais a serem resolvidas em cada etapa de recursão podem resultar em expressões extensas, impraticáveis para implementações e aplicações computacionais. Estas deficiências são contornadas pela presente metodologia, que contempla como procedimento de solução em cada etapa de recursão um método combinado de separação de variáveis juntamente com o princípio de Duhamel, onde a não linearidade aparece como heterogeneidade. Os termos fonte da equação em cada etapa da recursão são interpolados por polinômios e, utilizando a base de Gröbner dos pontos de ajuste, obtém-se o polinômio de grau reduzido para que a integração possa ser realizada facilmente. Como aplicação consideramos uma versão simplificada da equação de Navier-Stokes, que foi usada para simular o campo de vento fazendo uso dos dados micrometeorológicos do experimento de Copenhagen. A solução derivada foi avaliada em relação a estes dados experimentais dos experimentos de campo e mostrou que os resultados computados são aceitáveis e, portanto, a solução pode ser considerada aceitável e pode ser usada como um dispositivo de simulação para este tipo de experimentos de campo. Para quase todos os experimentos vinte autovalores e dez etapas de recursão foram suficientes. Como resultados, a velocidade do vento em determinadas posições foi simulada e comparada com os valores medidos. Os resultados obtidos permitem afirmar que a metodologia apresentada funciona satisfatoriamente e, portanto, pode ser considerada uma ferramenta promissora para resolução de problemas não lineares, que não são tratáveis com o método de decomposição convencional

Palavras-chave: Equação de Navier-Stokes; Base de Gröbner; Interpolação bidimensional; Polinômio de menor grau

1 INTRODUCTION

Solving nonlinear problems is still a challenge in science. Although several methods have been proposed some of them numerical others stochastic and only a few of them are based on analytical approaches such as the present one. One of the analytical methods is the so-called Adomian decomposition method, which was designed to solve in principle a variety of non-linear dynamical problems with several successful applications. A prominent feature of this method is that it makes use of a recursion prescription and further preserves the non-linearity without resorting to linearization or other simplifications, while providing an analytical expression as a solution. If the recursive scheme is convergent, then the method provides a solution, which in principal approaches the exact solution to any prescribed precision and is controlled by the depth of the recursive scheme, i.e. the number of differential equations to be solved in order to compose the final solution.

However, as the forthcoming discussion will show, even if the implementation is convergent, depending on the type of non-linearity, the source terms which contemplate the non-linearities that appear in each recursion step may be considerably extensive so that, even with programs that are suitable for symbolic manipulations, computer hardware imposes limits on the execution of advanced recursion steps. Due to this shortcoming one may either abandon the method, or as proposed in the present work resort to modifications and simplifications of the original decomposition scheme as proposed by Adomian. Although sacrificing exactness of the obtained solution in the present procedure, this should not be a final verdict for the modified method, as long as an approximate solution is close enough to the exact solution, by virtue of the fact that an adopted model, i.e. the partial differential equation that determines the dynamics of interest, is already an idealization with its associated model error.

A first attempt into this direction was a rather "brute force" approach as reported in Athayde et al. (2018). Although, application of the variable separation method together with Duhamel's Principle (Athayde et al. (2018), Duchateau and Zachmann (2002), Ozisik (1993)) in each recursion step should in principle make it feasible to compute the integrals of the inhomogeneous solution of this recursion, in practice this becomes increasingly tedious according to the combinatorial nature of constructing the Adomian polynomials for each new recursion step.

Nevertheless, the idea of constructing solutions of the inhomogeneous differential equations using a convolution of on the one hand a homogeneous solution composed by orthogonal functions and on the other hand the term from the inhomogeneity may be modified using the framework of the theory of Gröbner basis (Adams and Loustaunau (1994), Becker et al. (1993), Buchberger (1965), Farr and Gao (2006)). The scope of this theory is increasing and one can find applications in several areas, as shown for instance in references Berthomieu and Din (2022), Eder and Hofmann (2021), Liu et al. (2019), Bourgeois (2009), Zhang and Cheng (2004), Kwiecinsk (1991), Rabern (2007), Gerdet (1997), Hajnová and Pribylová (2019) and Gerdet (2004), where the majority refer to Symbolic Computation and Algebra.

Since the focus of this work is to provide expressions that may be implemented in a computationally efficient fashion (such as to allow for real time computing of flow fields) instead of using the rigor and functional considerations of Gröbner's mathematical theory, we determined the polynomials using a finite set of space points only. As a consequence, the solution constructed in such a way is no longer exact as the polynomials have the character of interpolating functions instead of being selected by mathematical orthogonality criteria in functional space. The latter is substituted by an optimization of polynomials to data calculated from the respective inhomogeneities in each recursion step.

A wind velocity profile in a planetary boundary layer scenario was chosen as an application and validated against the wind field data of the Copenhagen experiment.

2 A NAVIER-STOKES EQUATION BASED MODEL

Wind velocity fields in the planetary boundary layer are relevant in a variety of studies, from harnessing the wind up to the dispersion of substances by using the wind as a vehicle. Thus, as a starting point we consider the Navier-Stokes equation for a Newtonian, incompressible fluid and neglect external forces and the presence of pressure gradients, which shall describe the dominant dynamics of a velocity field $\vec{U}$ in domains with length scales of the order of $10^1\ km$

$$\rho \frac{D\vec{U}}{Dt} = \mu \nabla^2 \vec{U} \ , \tag{1}$$

where $\vec{U} = (u, v, w)^T$. In general one may choose the orientation of the coordinate system, which allows to align the dominant direction of the wind field with the x-axis. Then the equation for the dominant component u reads,

$$\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \ , \tag{2}$$

where $\nu = \mu/\rho$ (in units $[m^2/s]$) is the kinematic viscosity. Since in the boundary layer turbulent contributions are typically caused by thermal effects from vertical heat fluxes, we consider as a second spatial dimension the vertical coordinate z and render the model a two dimensional one. Further, in the equation above $\frac{\partial^2 u}{\partial y^2} \to 0$ is used.

The rectangular domain is defined by the dimensions $0 \leq x \leq L$, $0 \leq z \leq H$, the initial condition for the entering wind profile is represented by a known function $f(x, z)$

and at the domain boundaries the following conditions are assumed to hold.

$$u(x,z,0) = f(x,z), \tag{3}$$
$$\frac{\partial u}{\partial x}(0,z,t) = 0 \ , \tag{4}$$
$$\frac{\partial u}{\partial x}(L,z,t) = 0 \ , \tag{5}$$
$$\frac{\partial u}{\partial z}(x,0,t) = 0 \ , \tag{6}$$
$$\frac{\partial u}{\partial z}(x,H,t) = 0 \ . \tag{7}$$

Following now the prescription of Adomian's decomposition method the wind field is represented as the limit of a series $u(x,z,t) = \lim_{K\to\infty}\sum_{i=0}^{K} u_i(x,z,t)$, which upon insertion in Equation (2) results in one equation for $K+1$ unknown functions u_i,

$$\begin{aligned}\frac{\partial u_0}{\partial t} + \frac{\partial u_1}{\partial t} + \cdots + \frac{\partial u_K}{\partial t} + (u_0 + u_1 + \cdots + u_K)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial x} + \cdots + \frac{\partial u_K}{\partial x}\right) = \\ = \nu\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} + \cdots + \frac{\partial^2 u_K}{\partial x^2} + \frac{\partial^2 u_0}{\partial z^2} + \frac{\partial^2 u_1}{\partial z^2} + \cdots + \frac{\partial^2 u_K}{\partial z^2}\right) . \end{aligned} \tag{8}$$

The sub-determination of Equation (8) is now exploited to define the recursive scheme as the constitutive equation system for each component u_i. Note, that depending on the initial and boundary conditions restrictions the decomposition shall be setup such as to avoid the trivial solution. One of the possibilities for the recursive set of equations is,

$$\begin{aligned} i=0: &\ \frac{\partial u_0}{\partial t} - \nu\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial z^2}\right) = 0 \ , \\ i=1: &\ \frac{\partial u_1}{\partial t} - \nu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial z^2}\right) = -u_0\frac{\partial u_0}{\partial x} \ , \\ &\vdots \\ i=K: &\ \frac{\partial u_K}{\partial t} - \nu\left(\frac{\partial^2 u_K}{\partial x^2} + \frac{\partial^2 u_K}{\partial z^2}\right) = F_K(x,z,t) \ , \end{aligned} \tag{9}$$

where

$$F_K(x,z,t) = \underbrace{-u_{K-1}\frac{\partial u_{K-1}}{\partial x}}_{K>0} - \underbrace{\frac{\partial}{\partial x}\left(u_{K-1}\sum_{j=0}^{K-2} u_j\right)}_{K>1} ,$$

and the source term of each equation is formed from the nonlinear terms of the recursive equations whose solutions were already obtained in the previous recursion steps. By inspection one verifies that in the limit $K \to \infty$ all terms of Equation (8) are taken care of. Each of the equations in the recursive scheme have a unique solution, where the initial condition is satisfied already in the recursion initialization and all remaining initial conditions and all boundary conditions are homogeneous. Explicitly, conditions (3), (4) to (7) read,

$$u_i(x,z,0) = \delta_{i0} f(x,z) \ , \tag{10}$$
$$\frac{\partial u_i}{\partial x}(0,z,t) = 0 \ , \tag{11}$$
$$\frac{\partial u_i}{\partial x}(L,z,t) = 0 \ , \tag{12}$$
$$\frac{\partial u_i}{\partial z}(x,0,t) = 0 \ , \tag{13}$$
$$\frac{\partial u_i}{\partial z}(x,H,t) = 0 \ , \tag{14}$$

for $i \in \{0,\dots,K\}$ and δ_{i0} is the Kronecker symbol.

The solution of the recursion initialization (for u_0) is obtained by the well established variable separation method with $u_0(x,z,t) = X(x)Z(z)T(t)$,

$$u_0(x,z,t) = \sum_{m,n=0}^{\infty} A_{mn} e^{-\lambda_{mn}\nu t} \cos\left(\frac{m\pi}{H}z\right) \cos\left(\frac{n\pi}{L}x\right) \ , \tag{15}$$

with $m,n \in \mathbb{N}$, $\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2$,

$$A_{mn} = \frac{\kappa}{LH} \int_0^L \int_0^H f(x,z) \cos\left(\frac{m\pi}{H}z\right) \cos\left(\frac{n\pi}{L}x\right) \ dz\, dx \ , \tag{16}$$

and

$$\kappa = \begin{cases} 1 \text{ if } m = n = 0 \\ 2 \text{ if } mn = 0 \text{ and } m + n \neq 0 \\ 4 \text{ if } mn \neq 0 \end{cases} . \tag{17}$$

The differential equations for the recursion steps ($i \geq 1$) were solved using

Duhamel's principle (Duchateau and Zachmann (2002), Ozisik (1993)). This principle provides an analytical solution for inhomogeneous equations whose solution to the associated homogeneous problem is known. These solutions are given by

$$u_i(x,z,t) = \sum_{m,n=0}^{\infty} v_{mn}^{(i)}(t)\cos\left(\frac{m\pi}{H}z\right)\cos\left(\frac{n\pi}{L}x\right) \ ,$$

where the time dependent coefficients are

$$v_{mn}^{(i)} = \frac{\kappa}{LH}\int_0^t\int_0^L\int_0^H F_i(x,z,\tau)\cos\left(\frac{m\pi}{H}z\right)\cos\left(\frac{n\pi}{L}x\right)\ dz\, dx\ e^{\lambda_{mn}\nu(\tau-t)}\ d\tau\ . \tag{18}$$

A comment is in order here, by inspection of the recursion prescription (9) one observes the increase in terms composing the source term with each new recursion step. In cases where a higher number of recursions is necessary to attain a certain precision of the solution, this property can impose limits especially if automatization of this procedure is desired or needed.

Generically, the individual terms of the velocity field expansion $u_i(x,z,t)$ may be written as a product of two polynomials, one with the dependence on spatial variables and the other presenting the time evolution of the respective term

$$u_i(x,z,t) \approx P_i(x,z)T_i(t)\ . \tag{19}$$

A similar reasoning is applied to the source terms $F_i(x,z,t)$,

$$F_i(x,z,t) \approx R_i(x,z)S_i(t)\ . \tag{20}$$

In the next section, we will discuss how to determine the polynomials $P_i(x,z)$ and $R_i(x,z)$ using on the one hand some ideas of the theory of Gröbner basis together with a discrete set of source term points generated from the original expression, to be represented by the polynomials. The calculated data points of the source term open pathways to make use of interpolation algorithms, while theory of Gröbner basis allows to minimise the degree of the polynomials that represent the source terms in the solution procedure of each recursion step.

3 A GRÖBNER MODIFICATION TO THE ADOMIAN DECOMPOSITION METHOD

In this section we present the optimization procedure, which provides the polynomial with the smallest degree that represents best the source term for a finite set of spatial points. To this end we follow the reasoning of reference Farr and Gao (2006) and more detailed literature on the Gröbner basis formalism and some applications, which can be found in references Adams and Loustaunau (1994), Becker et al. (1993) and Buchberger (1965). Suppose, that a polynomial h with high degree describes satisfactorily the discrete set of points that characterize the source term in a specific recursion step. The objective is then to find the polynomial p with the smallest degree, that substitutes the source term with a certain fidelity.

Next, we will present part of the theory of Gröbner basis and, in particular, the algorithm that determines the reduced Gröbner basis from this discrete set of points, which renders executable the Adomian decomposition method for this type of non-linearity. To this end, consider for two spatial dimensions (x, z) the ring of polynomials with real coefficients $\mathbb{K} = \mathbb{R}[x, z]$, where for convenience one fixes a monomial ordering on $\mathbb{K}$. For a set of polynomials $M = \{m_1, \ldots, m_s\}$ with $M \subseteq \mathbb{K}$, consider the set

$$\mathcal{B}(M) = \{x^\alpha : \ \alpha \in \mathbb{N}^n \text{ and } LT(m_i) \nmid x^\alpha, 1 \le i \le s\} \ , \tag{21}$$

with x^α a monomial and $LT(m_i)$ the leading term of the polynomial m_i, that cannot be divided by $LT(m_i)$, for all m_i, with $1 \le i \le s$. Indivisibility $LT(m_i) \nmid x^\alpha$ implies x^α belongs to the set of monomials that form the rest of the division of a given polynomial by M.

Now, given the subset $S \subseteq \mathbb{K}$, one defines the variety, $V(S)$, in $\overline{\mathbb{K}}^n$ for the set of points P such that

$$V(S) = \{P \in \overline{\mathbb{K}}^n : \phi(P) = 0 \text{ for all } \phi \in S\} \ , \tag{22}$$

where $\overline{\mathbb{K}}$ is an extension of $\mathbb{K}$ in order to contain all the roots of polynomials with coefficients in $\mathbb{K}$. In the methodology presented in this work, $V(S)$ will be the set of points considered for the interpolation. Also this subset $V \subseteq \overline{\mathbb{K}}^n$ with ideal $I(V)$ in

$\mathbb{R}[x,z]$,

$$I(V) = \{\phi \in \mathbb{R}[x,z] : \phi(P) = 0 \text{ for all } P \in V\} \ , \tag{23}$$

is characterized by the set of distinct points $V = \{P_1, \ldots, P_m\}$ which are the common roots of the polynomials of $I(V)$. A set $G \subset I$ is a Gröbner basis of I if and only if $\|\mathcal{B}(G)\| = m$, that corresponds to the number of points P. In the methodology presented in this work, the polynomials of the set G will divide the polynomial h and the combination of the monomials of the set $\mathcal{B}(G)$, where the number of elements shall be equal to the number of points used in the interpolation, will then form the polynomial p.

Moreover, the Gröbner basis of the ideal of a finite set of points is constricted considering the points one by one, so that the polynomials obtained at each step always assume a zero value in all points already added. More specifically, for a finite set $V \subset \mathbb{K}$ with Gröbner basis $G = \{g_1, \ldots, g_t\}$ of ideal $I(V)$ and $P = (a_1, \ldots, a_n) \notin V$ then, if g_i is the polynomial in G that has the smallest leading term such that $g_i(P) \neq 0$, then the set

$$\widetilde{G} = \{\widetilde{g}_1, \ldots, \widetilde{g}_{i-1}, \widetilde{g}_{i+1}, \ldots, \widetilde{g}_t, g_{i1}, \ldots, g_{in}\}, \tag{24}$$

with

$$\widetilde{g}_j = g_j - \frac{g_j(P)}{g_i(P)} g_i, \ j \neq i \tag{25}$$

and

$$g_{ik} = (x_k - a_k) g_i, \ 1 \leq k \leq n, \tag{26}$$

is a Gröbner basis for $I(V \cup \{P\})$. Algorithm 1 shows the sequence of computational steps, which determine the reduced Gröbner basis of ideal $I(P_1, P_2, \ldots, P_m)$, where $N_G(\star)$ is the remainder of the division of $\star$ by G.

The smallest degree for a polynomial that represents the source term in each recursion step at a set of given spatial points $P_1, \ldots, P_m$ is

$$p(P_i) = r_i \ , \ 1 \leq i \leq m \ .$$

Algorithm 1 Algorithm to determine the reduced Gröbner basis of the ideal of a finite set of points (Farr and Gao (2006))

Require: $P_1, \ldots, P_m \in \mathbb{K}^n$ and fix the monomial order with increasing degree ($x_1 < x_2 < \cdots < x_n$)
Ensure: $G = \{g_1, \ldots, g_t\}$ the reduced Gröbner basis for $I(P_1, \ldots, P_m)$, in increasing order with respect to leading terms

```
G := {1} {the i-th polynomial in G is denoted g_i}
For k = 1, ..., m do
    Find the smallest i so that g_i(P_k) ≠ 0
    For j = i + 1, ..., ||G|| do
        g_j = g_j − (g_j(P_k) / g_i(P_k)) g_i
    End
    G := G\{g_i}
    For j = 1, ..., n do
        If x_j LT(g_i) not divisible by any LT of G then
            q = N_G((x_j − a_j) g_i)
            Insert q in ordered fashion into G
        End
    End
End
```

Following the reasoning of reference Farr and Gao (2006), once determined a polynomial $h \in \mathbb{K}$ such that $h(P_i) = r_i$, for $i = 1, \ldots, m$, then the polynomial p with smallest degree is obtained by determining $p = N_G(h)$ which still preserves equality at the spatial points $p(P_i) = r_i$ for $i = 1, \ldots, m$. The polynomial h may be constructed as follows. Recalling that the points $P_1, \ldots, P_m$ are mutually distinct one defines

$$h_{ij} = \frac{x_k - a_{jk}}{a_{ik} - a_{jk}} \ , \tag{27}$$

so that $h_{ij}(P_i) = 1$ and $h_{ij}(P_j) = 0$ for $i \neq j$. This guarantees that for each $i = 1, \ldots, m$, the polynomial

$$h_i = r_i \prod_{\substack{j=1 \\ j \neq i}}^{m} h_{ij} \ , \tag{28}$$

coincides with the desired values of the source term at a specific spatial point $h_i(P_i) = r_i$ and $h_i(P_j) = 0$, for all $j \neq i$. The polynomial that satisfies the desired properties is then $h = \sum_i h_i$ and the polynomial with the smallest degree passing through the specified spatial points P_i with value r_i is then given by $p = N_G(h)$, that is, the rest of the division of polynomial h by the Gröbner basis G polynomials. The Algorithm 2 summarizes this procedure.

Algorithm 2 Algorithm to determine the polynomial p with smallest degree

1: Determine the reduced Gröbner basis of the ideal generated by the points P_i with $i = 1, \ldots, m$ using Algorithm 1,
2: determine the polynomial h,
3: determine $p = N_G(h)$.

Step 3 of Algorithm 2 ensures that polynomial p assumes the same values as polynomial h at the points considered, since if p is the remainder of the division of h by the polynomials of G, then one can write h in the form $h = \omega_1 g_1 + \cdots + \omega_t g_t + p$, with $\omega_j \in \mathbb{K}$ and $1 \leq j \leq t$. For each point P considered $g_j(P) = 0$ holds and therefore $h(P) = p(P)$. Concerning the theory of Gröbner bases, these are the necessary tools that shall allow in the further to compute results that simulate the wind field of the Copenhagen experiment as obtained by the solution from the modified decomposition method using the parametrization as described in Equations (19) and (20).

4 VALIDATION OF THE TWO DIMENSIONAL MODEL AGAINST EXPERIMENTAL DATA

The modification of Adomian's decomposition method, i.e. substituting the source terms and velocity field expansion by the use of Gröbner's theory (see Equations (19) and (20)), was applied to observed data in the Copenhagen experiment. Although, the objective of this experiment was to provide data for substance dispersion modelling, some of the micrometeorological findings were used, i.e. the vertical wind field profile in the initial condition and the wind field evolution in the domain was simulated by the elaborated approximate solution of the Navier-Stokes equation. Details of the experiment may be found in references Gryning et al. (1987), Gryning (1981), Gryning and Lyck (1984) and Gryning and Lyck (2002), where for our purposes we extracted the wind field data at positions of interest. The relevant data of the experiment for fixing the vertical wind profile of the initial condition are given in Table 1, where $\overline{u}$ is the mean wind speed as measured at a certain height and mean signifies the time average over the measuring interval.

Further, in accordance with experimental observations, the initial condition was

Table 1 – Mean wind field data of the Copenhagen experiment for two heights, $z = 10\ m$ and $115\ m$, respectively

Experiments	Mean speed $\overline{u}$ (ms^{-1}) at $z = 10\ m$	Mean speed $\overline{u}$ (ms^{-1}) at $z = 115\ m$
Experiment 1 - 20 September	2.1	3.4
Experiment 2 - 26 September	4.9	10.6
Experiment 3 - 19 October	2.4	5.0
Experiment 4 - 03 November	2.5	4.6
Experiment 5 - 09 November	3.1	6.7
Experiment 6 - 30 April	7.2	13.2
Experiment 7 - 27 June	4.1	7.6
Experiment 8 - 06 July	4.2	9.4
Experiment 9 - 19 July	5.1	10.5

Source: Gryning et al. (1987), Gryning (1981), Gryning and Lyck (1984), Gryning and Lyck (2002)

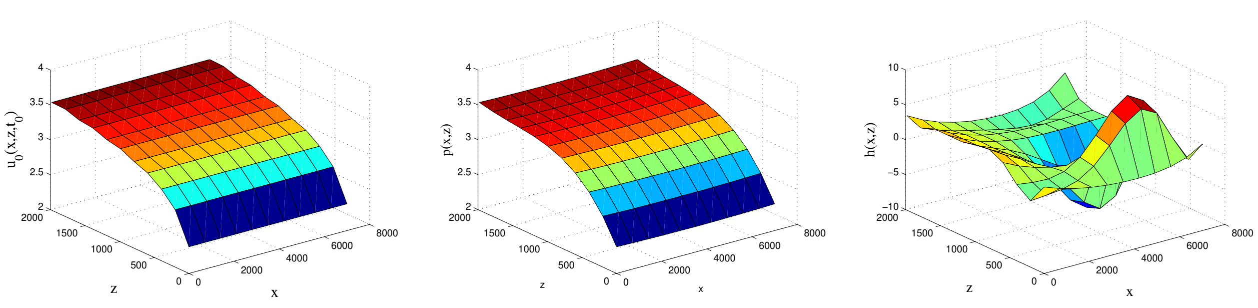
parametrized as follows,

$$u(x,z,0) = f(x,z) = \left(1 + b\cos\frac{x}{a}\right) U_r \left(\frac{z}{Z_r}\right)^{\eta}, \tag{29}$$

where η is a positive real number and varies between 0 and 1 (Irwin (1979)), $Z_r = 10\ m$ is a reference height where the wind speed U_r was measured during the execution of the experiment (Table 1). This initial condition is a power law profile with the addition of a disturbance in the variable x where the parameter b represents the amplitude of the disturbance, a its periodicity (Athayde et al. (2018)) and simulates a small variation of the velocity profile in the x direction. The numerical values of the remaining parameters considered for the simulation are: $\eta = 0.1$, $a = 3$ and $b = 0.01$.

The comparison for the time evolution of the term u_0 from the recursion initialization for $t_0 = 2177\ s$ to the optimized, i.e. the polynomial term p from the Gröbner formalism and a higher order polynomial h describing the same set of points as p, is shown in Figure 1, where the sum in Equation (15) was over ten eigenvalues for each spatial dimension.

Figure 1 – Initial condition $u_0(x,z,t_0)$, the optimized polynomial $p(x,z)$ of smallest degree and a $h(x,z)$ polynomial of higher degree for a set of eight spatial points



(a) $u_0(x,z,t_0)$ (b) $p(x,z)$ (c) $h(x,z)$

Source: the authors (2024)

To determine the polynomials p and h, respectively, the specific points in space for $u_0(x,z,t_0)$ are given below.

$$
\begin{aligned}
&x_0 = 3700, && z_0 = 115, && u_0(x_0,z_0,t_0)\\
&x_1 = 0, && z_1 = 1980, && u_0(x_1,z_1,t_0)\\
&x_2 = 7000, && z_2 = 0, && u_0(x_2,z_2,t_0)\\
&x_3 = 7000, && z_3 = 1980, && u_0(x_3,z_3,t_0)\\
&x_4 = 0, && z_4 = 0, && u_0(x_4,z_4,t_0)\\
&x_5 = 0, && z_5 = 594, && u_0(x_5,z_5,t_0)\\
&x_6 = 0, && z_6 = 1386, && u_0(x_6,z_6,t_0)\\
&x_7 = 0, && z_7 = 1584, && u_0(x_7,z_7,t_0)
\end{aligned}
\tag{30}
$$

Then, the explicit Gröbner basis G for this finite set of points is formed by the following polynomials,

$$g_1 = x^3 - 10700\ x^2 + 25900000\ x\ , \tag{31}$$

$$g_2 = zx^2 - 7000\ zx - 115\ x^2 + 805000\ x\ , \tag{32}$$

$$g_3 = z^2x - 1980\ zx - 64.992424242424590\ x^2 + 4.549469696969704 \times 10^5\ x\ , \tag{33}$$

$$
\begin{aligned}
g_4 = {} & z^5 - 5544\ z^4 + 11016324\ z^3 - 9.144097776 \times 10^9\ z^2 + \\
& - 2.235174179077148 \times 10^{-7}\ zx + 2.58208207488 \times 10^{12}\ z + \\
& + 1.570955461048121 \times 10^7\ x^2 - 1.099668822733682 \times 10^{11}\ x\ .
\end{aligned}
\tag{34}
$$

A polynomial h that passes through the same spatial points, but with degree larger than p, is given by

$$
\begin{aligned}
h = & -5.169433944689662 \times 10^{-45} z^7 x^7 + 7.237207522565526 \times 10^{-41} z^7 x^6 + \\
& - 2.533022632897933 \times 10^{-37} z^7 x^5 - 6.034856638502280 \times -32 z^7 x^3 + \\
& + 1.068169625014903 \times 10^{-27} z^7 x^2 - 6.083135491610295 \times 10^{-24} z^7 x + \\
& + 1.094119508560463 \times 10^{-20} z^7 + 3.889482099984502 \times 10^{-41} z^6 x^7 + \\
& - 5.444652939533052 \times 10^{-37} z^6 x^6 + 1.905616088827663 \times 10^{-33} z^6 x^5 + \\
& + 3.713312374954596 \times 10^{-28} z^6 x^3 - 6.572562903669636 \times 10^{-24} z^6 x^2 + \\
& + 3.743018873954233 \times 10^{-20} z^6 x - 6.732235335792683 \times 10^{-17} z^6 + \\
& - 1.136936559742312 \times 10^{-37} z^5 x^7 + 1.591302167643970 \times 10^{-33} z^5 x^6 + \\
& - 5.569475783554841 \times 10^{-30} z^5 x^5 - 8.255530580814937 \times 10^{-25} z^5 x^3 + \\
& + 1.461228912804243 \times 10^{-20} z^5 x^2 - 8.321574825461456 \times 10^{-17} z^5 x + \\
& + 1.496727694301747 \times 10^{-13} z^5 + 1.600271647147885 \times 10^{-34} z^4 x^7 + \\
& - 2.239267255246806 \times 10^{-30} z^4 x^6 + 7.837212783211774 \times 10^{-27} z^4 x^5 + \\
& + 7.181626257687949 \times 10^{-22} z^4 x^3 - 1.271147847610767 \times 10^{-17} z^4 x^2 + \\
& + 7.239079267749451 \times 10^{-14} z^4 x - 1.302028840518825 \times 10^{-10} z^4 + \\
& - 1.069421254107544 \times 10^{-31} z^3 x^7 + 1.495636759036660 \times 10^{-27} z^3 x^6 + \\
& - 5.234418057285528 \times 10^{-24} z^3 x^5 - 1.894666155459486 \times 10^{-20} z^3 x^3 + \\
& + 3.353559095163288 \times 10^{-16} z^3 x^2 - 1.909823484703163 \times 10^{-12} z^3 x + \\
& + 3.435029739848025 \times 10^{-09} z^3 + 2.642884739720159 \times 10^{-29} z^2 x^7 + \\
& - 3.688980907360812 \times 10^{-25} z^2 x^6 + 1.290922163011336 \times 10^{-21} z^2 x^5 + \\
& - 2.940656782108457 \times 10^{-16} z^2 x^3 + 5.204962504331970 \times 10^{-12} z^2 x^2 + \\
& - 2.964182036365324 \times 10^{-08} z^2 x + 5.331410745962633 \times 10^{-05} z^2 + \\
& - 3.408704151135992 \times 10^{-25} z x^6 + 1.261220535920317 \times 10^{-21} z x^5 + \\
& + 1.042718781790766 \times 10^{-13} z x^3 - 1.845612243769655 \times 10^{-09} z x^2 + \\
& + 1.051060532045092 \times 10^{-05} z x - 0.018904491513867 z + \\
& + 2.675157563463385 \times 10^{-23} x^6 - 9.898082984814524 \times 10^{-20} x^5 + \\
& - 8.183277022569647 \times 10^{-12} x^3 + 1.448440032994828 \times 10^{-07} x^2 + \\
& - 8.248743238750202 \times 10^{-04} x + 1.483628124191877 \ . \qquad (35)
\end{aligned}
$$

Upon executing the algorithm presented in Table 1 the smallest degree polynomial, obtained by $N_G(h)$ that results is

$$\begin{aligned} p = &-1.646204569424873\times10^{-14}z^4 + 2.403567130497672\times10^{-10}z^3 + \\ &- 8.589302313205728\times10^{-07}z^2 + 3.422180970202794\times10^{-12}zx + \\ &+ 0.001251849712247z + 8.749688551399750\times10^{-10}x^2 + \\ &- 6.110897644636018\times10^{-06}x + 1.483628124191877\ . \end{aligned} \quad (36)$$

Note that polynomial p is considerably simpler than polynomial h because of its smaller degree. The test that $G = \{g_1, g_2, g_3, g_4\}$ is really a Gröbner basis follows from $\mathcal{B}(G) = \{1, x, x^2, z, zx, z^2, z^3, z^4\}$ with $\|\mathcal{B}(G)\| = 8$ and the fact that eight spatial points were employed to determine p. Evidently, the monomials of $\mathcal{B}(G)$ are also monomials of p.

4.1 Results

To validate the model and verify if it is possible to reproduce a real situation, the methodology was used to simulate the Copenhagen experiment, obtaining the values for the mean wind. In Table 2 we show the results obtained in two dimensions. For the simulations 20 eigenvalues and 10 recursions were used, where $\overline{u}_o$ signifies the observed mean wind speed and $\overline{u}_p$ is the predicted mean wind speed from the simulations. The comparison shows that the predicted values for the mean wind speed are fairly close to the measured values by the experiment.

5 CONCLUSIONS

The developments of the present work focussed on a non-linear problem in form of a partial differential equation and a procedure, that leads to an approximate solution but in analytical representation. In the literature one finds the Adomian decomposition method, which is claimed to solve any non-linear problem whether deterministic or stochastic. Independent of the veracity of this statement in practical applications one faces the problem of increasingly extensive and complex expressions of the inhomogeneity of the equation in each recursion step, which turns it tedious if not impossible to determine the terms in each recursion which in the end composes

Table 2 – Comparison of the results with the Copenhagen experiments using an initial condition based on the mean wind speed in 10 meters, 20 eigenvalues and 10 recursions

Experiments	Distance (m)	Observed mean wind speed $\overline{u}_o$ (m/s)	Predicted mean wind speed $\overline{u}_p$ (m/s)
Experiment 1	1900	2.1	2.24
Experiment 1	3700	2.1	2.31
Experiment 2	2100	4.9	5.25
Experiment 2	4200	4.9	5.39
Experiment 3	1900	2.4	2.55
Experiment 3	3700	2.4	2.61
Experiment 3	5400	2.4	2.61
Experiment 4	4000	2.5	2.74
Experiment 5	2100	3.1	3.35
Experiment 5	4200	3.1	3.38
Experiment 5	6100	3.1	3.34
Experiment 6	2000	7.2	7.61
Experiment 6	4200	7.2	7.82
Experiment 6	5900	7.2	7.69
Experiment 7	2000	4.1	4.37
Experiment 7	4100	4.1	4.50
Experiment 7	5300	4.1	4.48
Experiment 8	1900	4.2	4.51
Experiment 8	3600	4.2	4.55
Experiment 8	5300	4.2	4.58
Experiment 9	2100	5.1	5.51
Experiment 9	4200	5.1	5.64
Experiment 9	6000	5.1	5.54

Source: the authors (2024)

the solution of the problem. In this discussion we considered a simplified version of the Navier-Stokes equation with its non-linear term by the velocity field times its gradient.

In order to circumvent the afore mentioned laboriousness of the traditional decomposition method a new methodology was presented in order to resuscitate the method but in modified fashion. The elaborated procedure leads to an approximate solution in form of an analytical expression, where the variable separation method in the spatial part together with Duhamel's principle for the time dependence was applied in each recursion step to solve the respective inhomogeneous differential equations. The key for rendering the decomposition method tractable is based on the approximation of the non-linear terms which appear as the inhomogeneity of the recursive equations. Recalling, that the original decomposition method constructs the inhomogeneity of the differential equations using the solutions of the previous recursion steps so that these are known. This allows to represent the inhomogeneity by a discrete set of coordinates, which are interpolated by a polynomial. One the one hand polynomials simplify integration but on the other hand invoke oscillations which in general introduce significant deviations of the formally determined solution and the exact solution. The use of a Gröbner basis and its reduction to the remainder of the division of any polynomial that represents the set of points for which the inhomogeneity is calculated leads to the smallest polynomial that best represents the inhomogeneity and also minimizes possible oscillations in the resulting interpolation of the points.

Evidently a solution constructed in such a way is at best an approximate solution and, therefore, it is important to use experimental data or reference data to validate it. To this end the micrometeorological data of the Copenhagen experiment were selected and the wind field was simulated at the locations, where the wind speed was measured. Most of the experimental findings could be described with fairly good agreement using 20 eigenvalues and 10 recursion steps, which lead to an analytical expression and allows for computational simulations in real time – a useful feature for this type of field experiments.

Nevertheless, the authors believe that with the novel method we opened pathways for applications of the modified decomposition method to other non-linear problems, where we expect that an approximate solution may be determined in analytical representation reflecting the characteristic dynamical features of the model in consideration, i.e. the underlying partial differential equation.

REFERENCES

Adams, W. W. and Loustaunau, P., editors (1994). *An Introduction to Grobner Bases*. American Mathematical Society, United State of America.

Athayde, A. S., Piovesan, L. R., Bodmann, B. E. J., and Vilhena, M. T. M. B. (2018). Analytical solution of the coupled advection-diffusion and Navier-Stokes equation for air pollutant emission simulation. *American Journal of Environmental Engineering*, 8(4):150–153.

Becker, T., Weispfenning, V., and Kredel, H., editors (1993). *Grobner Bases. A*

Computational Approach to Commutative Algebra. Springer Verlag, New York.

Berthomieu, J. and Din, M. S. E. (2022). Guessing Grobner bases of structured ideals of relations of sequences. *Journal of Symbolic Computation*, 111:1–26.

Bourgeois, G. (2009). Algebraic systems of matrices and Grobner basis theory. *Linear Algebra and its Applications*, 430:2157–2169.

Buchberger, B. (1965). An algorithm for finding the basis elements in the residue class ring modulo a zero dimensional polynomial ideal. PhD thesis presented at the Mathematical Institute, University of Innsbruck, Austria.

Duchateau, P. and Zachmann, D. (2002). *Applied Partial Differential Equation*. Dover Publications, New York.

Eder, C. and Hofmann, T. (2021). Efficient Grobner bases computation over principal ideal ring. *Journal of Symbolic Computation*, 103:1–13.

Farr, J. B. and Gao, S. (2006). Computing grobner bases for vanishing ideals of finite sets of points. International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes, Las Vegas, NV, USA, 20–24 February 2006, LNCS 3857, pages 118-127.

Gerdet, V. P. (1997). Grobner bases and involutive methods for algebraic and differential equations. *Mathematical and Computer Modelling*, 25:75–90.

Gerdet, V. P. (2004). Grobner bases in perturnative calculations. *Nuclear Physics B*, 135:232–237.

Gryning, S., Holtslag, A., Irwing, J., and Silversten, B. (1987). Applied dispersion modelling based on meteorological scaling parameters. *Atmospheric Environment*, 21(1):79–89.

Gryning, S. E. (1981). Elevated source SF6 – tracer dispersion experiments in the Copenhagen area. *Report RISOE–R–446, Risoc National Laboratory, Roskilde, Denmark*.

Gryning, S. E. and Lyck, E. (1984). Atmospheric dispersion from elevated source in an urban area: comparison between tracer experiments and model calculations. *J. Appl. Meteor.*, 23:651–654.

Gryning, S. E. and Lyck, E. (2002). The Copenhagen tracer experiments: reporting of measurements. *Riso National Laboratory*.

Hajnová, V. and Pribylová, L. (2019). Bifurcation manifolds in predator-prey models computed by Grobner basis method. *Mathematical Biosciences*, 312:1–7.

Irwin, J. S. (1979). Estimating plume dispersion - a recommended generalized scheme. *Proceeding of the Fourth Symposium on Turbulence, Diffusion and Air Pollution - AMS*, pages 62–69.

Kwiecinsk, M. (1991). A Grobner basis criterion for isomorphisms of algebraic varieties. *Journal of Pure ans Applied Algebra*, 74:275–279.

Liu, J., Li, D., and Liu, W. (2019). Some criteria for Grobner bases and their applications. *Journal of Symbolic Computation*, 92:15–21.

Ozisik, M. (1993). *Heat Conduction*. John Wiley & Sons, New York, 2 edition.

Rabern, L. (2007). Applying Grobner basis techniques to group theory. *Journal of Pure and Applied Algebra*, 210:137–140.

Zhang, S. and Cheng, S. S. (2004). Grobner basis for an ideal of a polynomial ring over an algebraic extension over a field and its applications. *Applied Mathematics and Computation*, 153:27–58.

Author contributions

1 – Luciana Rossato Piovesan (Corresponding Author)
Graduated in Mathematics, Master's in Applied Mathematics and PhD in Mechanical Engineering.
https://orcid.org/0000-0001-6704-7522 • lurpiovesan@gmail.com
Contribution: Conceptualization, Formal Analysis, Methodology, Software, Supervision, Validation, Writing – original draft, Writing – review & editing

2 – Alexandre Sacco de Athayde
Graduated in Mathematics, Master's in Applied Mathematics and PhD in Mechanical Engineering.
https://orcid.org/0000-0003-2874-3685 • alexandre.athayde@ufpel.edu.br
Contribution: Conceptualization, Formal Analysis, Methodology, Software, Supervision, Validation, Writing – original draft, Writing – review & editing

3 – Bardo Ernst Josef Bodmann
Graduated, Master's and PhD in Physics.
https://orcid.org/0000-0003-1759-5525 • bbod@gmx.com
Contribution: Conceptualization, Formal Analysis, Methodology, Software, Supervision, Validation, Writing – original draft, Writing – review & editing

4 – Marco Túllio Menna Barreto de Vilhena
Graduated in Civil Engineering, Master's in Mathematics and PhD in Mechanical Engineering.
https://orcid.org/0000-0003-2427-5175 • mtmbvilhena@gmail.com
Contribution: Conceptualization, Formal Analysis, Methodology, Software, Supervision, Validation, Writing – original draft, Writing – review & editing

How to cite this article

Piovesan, L. R., Atahyde, A. S., Bodmann, B. E. J., & Vilhena, M. T. M. B. (2024). A solution to the time-dependent two-dimensional Navier-Stokes equation in a rectangular domain using the Adomian decomposition method and theory of Gröbner Basis. *Ciência e Natura*, Santa Maria, v. 46, spe. 1, e87459. https://doi.org/10.5902/2179460X87459