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Mathematics

Asymptotic homogenization of a problem for a wave equation on a microperiodic medium

Homogeneização assintótica de um problema para uma equação de onda sobre um meio microperiódico

Douglas Machado da Silva [I], **Leslie Darién Pérez-Fernández** [I], **Alexandre Molter** [I], **Julián Bravo-Castillero** [II]

[I] Institute of Physics and Mathematics, Federal University of Pelotas, Pelotas, RS, Brazil
[II] Institute for Research in Applied Mathematics and Systems, National Autonomous University of Mexico, Mérida, Mexico

ABSTRACT

The asymptotic homogenization method is a mathematical technique that allows studying the physical properties of a micro-heterogeneous, periodic medium characterized by rapidly oscillating coefficients through a homogeneous medium that is asymptotically equivalent to the micro-heterogeneous medium. The method involves constructing a two-scale formal asymptotic solution of the original problem, and by applying mathematical formalism, a problem is formulated over a homogenized medium known as the homogenized problem. This work aims to apply this method to a problem for a hyperbolic equation and demonstrate the proximity between the solutions.

Keywords: Wave equation; Formal asymptotic solution; Asymptotic homogenization method

RESUMO

O metódo de homogeneização assintótica trata-se de uma técnica matemática que permite estudar as propriedades físicas de um meio micro-heterogêneo, periódico e caracterizado por coeficientes rapidamentes oscilantes, através de um meio homogêneo que é assintoticamente equivalente ao meio micro-heterogêneo. O método consiste em construir uma solução assintótica formal em duas escalas do problema original e, ao aplicar o formalismo matemático, constrõe-se um problema sobre um meio homogêneo chamado de problema homogeneizado. O presente trabalho tem como objetivo aplicar este método a um problema para uma equação hiperbólica e demonstrar a proximidade entre as soluções.

Palavras-chave: Equação da onda; Solução assintótica formal; Método de homogeneização assintótica

1 INTRODUCTION

A heterogeneous medium is characterized by the variation of its physical properties throughout its structure. In particular, a periodic medium is characterized by the composition of its structure through the periodic reproduction of a recurring element called the basic (or periodicity) cell. In nature, there are several examples of heterogeneous materials, such as bone and soil. We also find examples for manufactured materials, such as ceramics and gels (Torquato, 2002). Therefore, the importance of studying such materials is evident, especially how the interaction between their physical and geometric properties occurs. One way to approach this is through the use of the so-called asymptotic homogenization method (AHM)(Bakhvalov and Panasenko, 1989; Bensoussan et al., 1978; Ciouranescu and Donato, 2000; Tartar, 2009), which is a mathematical tool that allows us to evaluate the physical behavior of a periodic micro-heterogeneous medium, whose separation of structural scales is characterized by the small parameter $\varepsilon > 0$. The AHM obtains a two-scale formal asymptotic solution (FAS) of the exact solution u^ε of the original problem via the solution u_0 of the problem over the equivalent homogeneous medium and the local problems over the periodic cell. The justification of AHM, i.e., that u_0 is a good approximation of u^ε, consists of showing, using the estimate resulting from some maximum principle suitable for each case, that u_0 is an asymptotic expansion (AE) of u^ε with respect to the norm of the space in which they are sought, i.e., that $u^\varepsilon - u_0 = O(\varepsilon^M)$, for some $M \in \mathbb{R}_+^*$, as $\varepsilon \to 0^+$.

And given the versatility and effectiveness of this method, examples of the use of homogenization theory can be found in various areas, for example: topological optimization (Bendsøe and Sigmund, 2003; Da, 2019), optimal material design (Ciblac and Morel, 2014), bone biomechanics (Parnel and Grimal, 2009), structural failure prediction (Pérez-Fernández and Beck, 2014), seismic wave propagation (Capdville et al., 010a,0), nuclear reactor physics (Allaire and Bal, 1999; Weston, 2007), transport of a chemical species (Ng, 2006), fluid mechanics (Dimitrienko, 1997), fracture mechanics (Dormieux and Kondo, 2016), coupled phenomena analysis (Auriault et al., 2009), mathematical medicine (Desbrun et al., 2013), and pollutant dispersion (Costa et al., 2018).

In this current work, we will address the application of the AHM to a problem for the wave equation, considering elastic effects. Finally, we will present the demonstration of the proximity between the solutions of the original and homogenized problems. This work constitutes an introductory study of wave phenomena in microstructured media, but differs from other works in the field both in its different approaches and in its richness in details, as it encompasses all the fundamental steps of AHM, along with an example, which is also presented in full detail.

2 METHODOLOGY

2.1 Preliminaries

Lemma: Let $F(y)$ e $a(y)$ be 1-periodic differentiable functions, with $a(y)$ strictly positive and bounded. A necessary and sufficient condition for a 1-periodic solution $N(y)$ of the equation $\mathcal{L}N = F$, with $\mathcal{L} \equiv \frac{d}{dy}\left(a(y)\frac{d}{dy}\right)$, to exist is that $\langle F \rangle \equiv \int_0^1 F(y)dy = 0$. In addition, the solution $N(y)$ is unique up to an additive constant, that is, $N(y, C) = \tilde{N}(y) + C$, where $\tilde{N}(0) = 0$ and C is a constant (Bakhvalov and Panasenko, 1989).

Generalized Maximum Principle for Hyperbolic Equations: Let the generalized solution $u \in H_0^1((0,1) \times (0,T))$, $(0,1) \subset \mathbb{R}^d$, of the problem

$$\begin{cases} R(x,t)\dfrac{\partial^2 u}{\partial t^2} + S(x,t)\dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x_i}\left(A_{ij}(x)\dfrac{\partial u}{\partial x_j}\right) \\ \quad + B_i(x,t)\dfrac{\partial u}{\partial x_i} + A(x,t)u = f(x,t) + \dfrac{\partial f_i(x,t)}{\partial x_i} \; , \; x \in (0,1) \setminus \Gamma \, , \; t \in (0,T) \\ u\big|_{\partial(0,1)} = 0 \, , \; t \in (0,T) \\ u(x,0) = \psi_1(x) \, , \; \psi_1(x) \in H_0^1((0,1)) \\ \dfrac{\partial u}{\partial t}(x,0) = \psi_2(x) \, , \; \psi_2 \in L^2((0,1)) \end{cases} ,$$

where $R(x,t) > R_0 > 0$, $S(x,t) > S_0 > 0$, R_0 and S_0 are constants. The coefficients A_{ij} satisfy the symmetry conditions $A_{ij}(x) = A_{ji}(x)$ and positive definiteness $A_{ij}\eta_i\eta_j \geq k\eta_i\eta_j$, $\forall \eta \in \mathbb{R}^d$, with $k > 0$ constant. For the solution u, the following estimate is valid

$$\|u\|_{H_0^1((0,1)\times(0,T))} \leq c\Bigg(\|\psi_1(x)\|_{H_0^1((0,1))} + \|\psi_2(x)\|_{L^2((0,1))} + \|f\|_{L^2((0,1)\times(0,T))} + \sum_{i=1}^{s}\left(\|f_i\|_{L^2((0,1)\times(0,T))} + \max_{t\in(0,T)} \|f_i\|_{L^2((0,1))} \right)\Bigg), \quad (1)$$

where c is dependent of T (Bakhvalov and Panasenko, 1989).

2.2 Problem formulation

Consider the problem of mechanical vibrations of a microperiodic string with Young's modulus $a\,(x/\varepsilon)$, clamped of both ends with initial displacement $p(x)$, initial velocity $q(x)$ and body force $f(x,t)$, where u^ε is the vertical displacement. Furthermore, it follows that a, p, q and f are differentiable functions and a is ε-periodic, strictly positive and bounded.

This problem is modeled as for each ε, $0 < \varepsilon \ll 1$, find $u^\varepsilon \in C^2((0,1)\times(0,T)) \cap C^1([0,1]\times[0,T])$, solution of the wave equation

$$L^\varepsilon u^\varepsilon \equiv \frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{\partial}{\partial x}\left(a^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x}\right) = f(x,t),\; x \in (0,1) \quad (2)$$

subject to the boundary conditions

$$u^\varepsilon(0,t) = u^\varepsilon(1,t) = 0\ ,\ t \in (0,T)\,, \quad (3)$$

and the initial conditions

$$u^\varepsilon(x,0) = p(x)\,,\ \frac{\partial u^\varepsilon}{\partial t}(x,0) = q(x)\ ,\ x \in (0,1)\,, \quad (4)$$

where the compatibility conditions $p(0) = p(1) = 0$ must be satisfied.

2.3 AHM aplication

A FAS of the problem defined by Eqs. (2)-(4) is sought as the following asymptotic expansion of the exact solution $u^\varepsilon(x)$:

$$u^{(2)}(x,t,\varepsilon) = u_0(x,y,t) + \varepsilon u_1(x,y,t) + \varepsilon^2 u_2(x,y,t)\ ,\ y = \frac{x}{\varepsilon}\,,\ \varepsilon = \frac{1}{n}\,,\ n \in \mathbb{N}\,, \quad (5)$$

where unknown functions u_k, $k \in \{0, 1, 2\}$, are twice continually differentiable in x, y and t, and 1-periodic functions in y.

we have, by the chain rule

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial x} + y_x \frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial y} \ . \tag{6}$$

By substituting Eq. (6) into (2), we get

$$\left(\frac{\partial^2}{\partial t^2} - \mathcal{L}_{xx} - \varepsilon^{-1} \big(\mathcal{L}_{xy} + \mathcal{L}_{yx} \big) - \varepsilon^{-2} \mathcal{L}_{yy} \right) u^{\varepsilon} = f(x, t) \ , \tag{7}$$

where the linear operators $\mathcal{L}_{\alpha\beta}$, $\alpha, \beta \in \{x, y\}$, are defined as

$$\mathcal{L}_{\alpha\beta} = \frac{\partial}{\partial \alpha} \left(a(y) \frac{\partial}{\partial \beta} \right) \ , \ \alpha, \beta \in \{x, y\} \ .$$

By substituting Eq. (5) into Eq. (7),we obtain

$$\left(\frac{\partial^2}{\partial t^2} - \mathcal{L}_{xx} - \varepsilon^{-1} \big(\mathcal{L}_{xy} + \mathcal{L}_{yx} \big) - \varepsilon^{-2} \mathcal{L}_{yy} \right) (u_0 + \varepsilon u_1 + \varepsilon^2 u_2) = f(x, t) \ . \tag{8}$$

Rearranging the terms by powers of ε into Eq. (8) , it is obtained that

$$\begin{aligned} &(-\mathcal{L}_{yy} u_0) \varepsilon^{-2} + (-\mathcal{L}_{xy} u_0 - \mathcal{L}_{yx} u_0 - \mathcal{L}_{yy} u_1) \varepsilon^{-1} \\ &\qquad + \left(\frac{\partial^2 u_0}{\partial t^2} - \mathcal{L}_{xx} u_0 - \mathcal{L}_{xy} u_1 - \mathcal{L}_{yx} u_1 - \mathcal{L}_{yy} u_2 - f(x, t) \right) \varepsilon^0 = O(\varepsilon) \ . \end{aligned}$$

So, in order to ensure the existence of the FAS in Eq. (5) of the problem in Eqs. (2)-(4), it is necessary to ensury the existence of solutions u_k, $k \in \{0, 1, 2\}$, 1-periodic in y for the following recurrence of differential equations for $(x, y, t) \in (0, 1) \times (0, n) \times (0, T)$:

$$\begin{cases} \varepsilon^{-2} : \mathcal{L}_{yy} u_0 = 0 \,, \\ \varepsilon^{-1} : \mathcal{L}_{yy} u_1 = -\mathcal{L}_{xy} u_0 - \mathcal{L}_{yx} u_0 \,, \\ \ \varepsilon^{0} : \mathcal{L}_{yy} u_2 = \dfrac{\partial^2 u_0}{\partial t^2} - \mathcal{L}_{xx} u_0 - \mathcal{L}_{xy} u_1 - \mathcal{L}_{yx} u_1 - f(x, t) \,. \end{cases} \tag{9}$$

Note that the differential equations in Eq. (9) are of the form $\mathcal{L}_{yy} u_k = F$. Thus, considering x and y mutually independent variables, we can use the lemma in Subsection 2.1, for each x fixed, taking $\mathcal{L} \equiv \mathcal{L}_{yy}$ and $N \equiv u_k$.

Applying the Lemma for the first equation in Eq. (9) with $N \equiv u_0$ and $F \equiv 0$, it can be concluded that the existence of solution u_0 1-periodic in y is guaranteed and also u_0 does not depend on y, i.e., $u_0 = u_0(x)$.

So, the second differential equation in Eq. (9) is rewritten as

$$\mathcal{L}_{yy} u_1 = -\frac{da}{dy}\frac{\partial u_0}{\partial x}. \tag{10}$$

By applying the Lemma, it can be concluded that the existence of solution u_1 1-periodic in y is guaranteed, observing that

$$\left\langle -\frac{da}{dy}\frac{\partial u_0}{\partial x} \right\rangle = -\frac{\partial u_0}{\partial x}\int_0^1 \frac{da}{dy} dy = 0\,,$$

due to the 1-periodicity of $a(y)$ inherited from ε-periodicity of $a\left(\frac{x}{\varepsilon}\right)$. Then, the structure of the right-hand side of Eq. (10), we suppose

$$u_1(x, y, t) = N_1(y)\frac{\partial u_0}{\partial x}\ , \tag{11}$$

where N_1 is 1-periodic.

Note that by substituting Eq. (11) into Eq. (10) and assuming that $\dfrac{\partial u_0}{\partial x} \neq 0$, it is obtained that N_1 is the 1-periodic solution of the so-called *first local problem* defined by the equation

$$\frac{d}{dy}\left(a(y) + a(y)\frac{dN_1}{dy}\right) = 0\,, \tag{12}$$

subject to the condition $N_1(0) = 0$.

So it can be concluded that

$$N_1(y) = \int_0^y \left(\frac{\widehat{a}}{a(s)} - 1\right) ds\,, \tag{13}$$

where $\widehat{a} = \langle (a(y))^{-1}\rangle^{-1}$ is the so-called *effective coefficient*.

Finally, by applying the Lemma to the third equation in Eq. (9), using the independence of u_0 with respect to y and taking into account Eq. (11), we obtain that the condition for the existence of a 1-periodic solution u_2 in y is that u_0 should be the solution of the so-called *homogenized problem* defined by the equation

$$\mathcal{L}^0 u_0 \equiv \frac{\partial^2 u_0}{\partial t^2} - \widehat{a}\frac{\partial^2 u_0}{\partial x^2} = f(x,t)\,, \tag{14}$$

subject to the initial and boundary conditions obtained by substituting the FAS in Eq.(5), the conditions of the original problem.

Therefore, considering u_0 as the solution of the homogenized problem, we have that the equation for u_2 in (9) can be rewritten as

$$\mathcal{L}_{yy} u_2 = -\frac{d}{dy}\left(a(y)N_1(y)\right)\frac{\partial^2 u_0}{\partial x^2}\ , \tag{15}$$

for which, considering the structure of the right-hand side, we assume that

$$u_2(x,y,t) = N_2(y)\frac{\partial^2 u_0}{\partial x^2}\,, \tag{16}$$

where N_2 is 1-periodic.

Substituting Eq. (16) into Eq. (15) and assuming that $\dfrac{\partial^2 u_0}{\partial x^2} \neq 0$, it is obtained that N_2 is the 1-periodic solution of the so-called *second local problem* defined by the equation

$$\frac{d}{dy}\left(a(y)N_1(y) + a(y)\frac{dN_2}{dy}\right) = 0\,, \tag{17}$$

subject to the condition $N_2(0) = 0$, so it can be concluded that

$$N_2(y) = \int_0^y \left(\frac{\widehat{a}\langle N_1(y)\rangle}{a(s)} - N_1(s)\right) ds\,. \tag{18}$$

Therefore, from Eqs. (5), (11) and (16), we have the follow expression for the FAS

$$u^{(2)}(x,t,\varepsilon) = u_0(x,t) + \varepsilon N_1\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x} + \varepsilon^2 N_2\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0}{\partial x^2}\ . \tag{19}$$

So, the FAS for the problem in Eqs. (2)-(4) is given by Eq. (19), in which N_1 and N_2 are given by Eqs.(13) and (18), respectively, and u_0 is the solution of the homogenized problem, defined by

$$\begin{cases} \dfrac{\partial^2 u_0}{\partial t^2} - \widehat{a}\dfrac{\partial^2 u_0}{\partial x^2} = f(x,t), \quad x \in (0,1)\,,\, t > 0 \\ u_0(0,t) = u_0(1,t) = 0 \\ u_0(x,0) = p(x)\,,\, \dfrac{\partial u_0}{\partial t}(x,0) = q(x) \end{cases} . \tag{20}$$

2.4 Proximity relation

To demonstrate the proximity between the solutions u^ε of the original problem and u_0 of the homogenized problem, we use the maximum principle for hyperbolic equations in Subsection 2.1 (Bakhvalov and Panasenko, 1989). However, we cannot consider the estimate in Eq. (1) immediately, as the problems were defined in terms of different operators, $\mathcal{L}^\varepsilon$ and $\mathcal{L}^0$, respectively, according to Eqs. (2) and (14). To overcome this, we consider an auxiliary problem based on the FAS

$$u^{(1)}(x,t,\varepsilon) = u_0(x,t) + \varepsilon N_1\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x}\ , \tag{21}$$

obtained from Eq. (19), considering $N_2 \equiv 0$.

By using Eqs. (2), (12), (14) and (17) and considering $y = \dfrac{x}{\varepsilon}$, we obtain that the expression for the error $\mathcal{L}^\varepsilon u^{(1)} - f$ committed when approximating the exact solution u^ε of the original problem in Eqs. (2)-(4) with the FAS $u^{(1)}$ in Eq. (21) is given by

$$\mathcal{L}^\varepsilon u^{(1)} - f(x,t) = \varepsilon\left(N_1(y)\frac{\partial^3 u_0}{\partial t^2 \partial x} - N_1(y)a(y)\frac{\partial^3 u_0}{\partial x^3}\right) \equiv F^\varepsilon,\ (x,t) \in (0,1)\times(0,T). \tag{22}$$

Evaluating Eq. (21) at $x \in \{0,1\}$ and considering Eq. (20), we obtain the boundary and initial conditions for the FAS $u^{(1)}$:

$$u^{(1)}(0,t,\varepsilon) = u^{(1)}(1,t,\varepsilon) = 0, \quad u^{(1)}(x,0,\varepsilon) = p(x), \quad \frac{\partial u^{(1)}}{\partial t}(x,0,\varepsilon) = q(x). \tag{23}$$

Therefore, it is obtained the following problem for the FAS $u^{(1)}$ defined in terms of the operator $\mathcal{L}^\varepsilon$ of the original problem in Eqs. (2)-(4):

$$\begin{cases} \mathcal{L}^{\varepsilon} u^{(1)} = F^{\varepsilon} + f(x,t), \quad x \in (0,1)\,,\, t>0 \\ u^{(1)}(0,t,\varepsilon) = u^{(1)}(1,t,\varepsilon) = 0 \\ u^{(1)}(x,0,\varepsilon) = p(x),\ \dfrac{\partial u^{(1)}}{\partial t}(x,0,\varepsilon) = q(x) \end{cases} . \tag{24}$$

Thus, note that by subtracting the problem in Eq. (24) from the original problem in Eqs.(2)-(4), we obtain the following problem:

$$\begin{cases} \mathcal{L}^{\varepsilon}(u^{\varepsilon} - u^{(1)}) = -F^{\varepsilon}, \quad x \in (0,1)\,,\, t>0 \\ u^{\varepsilon}(0,t) - u^{(1)}(0,t,\varepsilon) = u^{\varepsilon}(1,t) - u^{(1)}(1,t,\varepsilon) = 0 \\ u^{\varepsilon}(x,0) - u^{(1)}(x,0,\varepsilon) = 0, \quad \dfrac{\partial u^{\varepsilon}}{\partial t}(x,0) - \dfrac{\partial u^{(1)}}{\partial t}(x,0,\varepsilon) = 0 \end{cases} . \tag{25}$$

Applying the estimative in Eq.(1) in the problem into Eq. (25), we have the following estimate for $u^{\varepsilon} - u^{(1)}$:

$$\|u^{\varepsilon} - u^{(1)}\|_{H_0^1((0,1)\times(0,T))} \leq c(T) \|F^{\varepsilon}\|_{L^2((0,1)\times(0,T))}\ . \tag{26}$$

As we use the norm of the space $L^2((0,1)\times(0,T))$, we have from Eq. (22) that

$$\|F^{\varepsilon}\|^2_{L^2((0,1)\times(0,T))} = \varepsilon^2 \int_0^T \int_0^1 \left(N_1(y) \frac{\partial^3 u_0}{\partial t^2 \partial x} - N_1(y) a(y) \frac{\partial^3 u_0}{\partial x^3} \right)^2 dx\, dt\ . \tag{27}$$

Moreover, we have that

$$\begin{aligned} \left(N_1(y) \frac{\partial^3 u_0}{\partial t^2 \partial x} - N_1(y) a(y) \frac{\partial^3 u_0}{\partial x^3} \right)^2 &= \left| N_1(y) \frac{\partial^3 u_0}{\partial t^2 \partial x} - N_1(y) a(y) \frac{\partial^3 u_0}{\partial x^3} \right|^2 \\ &\leq \left(\left| N_1(y) \frac{\partial^3 u_0}{\partial t^2 \partial x} \right| + \left| N_1(y) a(y) \frac{\partial^3 u_0}{\partial x^3} \right| \right)^2\ . \end{aligned}$$

Assuming that $u_0(x,t) \in C^3([0,1])$ for all $t \in [0,T]$ and that $u_0(x,t) \in C^2([0,T])$ for all $x \in [0,1]$, it follows from the Weierstrass Theorem (Lima, 2018) that there exist constants $A_1, A_2 > 0$, such that

$$\left| \frac{\partial^3 u_0}{\partial t^2 \partial x} \right| \leq A_1\ ,\ \left| \frac{\partial^3 u_0}{\partial x^3} \right| \leq A_2\ .$$

Then, it follows that

$$\left(N_1(y)\frac{\partial^3 u_0}{\partial t^2 \partial x} - N_1(y)a(y)\frac{\partial^3 u_0}{\partial x^3}\right)^2 \leq A^2 N_1^2(y)\left(a(y)+1\right)^2 , \tag{28}$$

where $A = \max\{A_1, A_2\}$.

From Eqs. (27) e (28), we obtain

$$\begin{aligned} \|F^\varepsilon\|^2_{L^2((0,1)\times(0,T))} &\leq \varepsilon^2 A^2 \int_0^T \int_0^1 N_1^2(y)\big(a(y)+1\big)^2 dx \\ &= \varepsilon^2 A^2 T \int_0^1 N_1^2(y)\big(a(y)+1\big)^2 dx \\ &= \varepsilon^2 A^2 T \int_0^1 N_1^2\left(\frac{x}{\varepsilon}\right)\left(a\left(\frac{x}{\varepsilon}\right)+1\right)^2 dx . \end{aligned} \tag{29}$$

Being $N_1\left(\frac{x}{\varepsilon}\right)$, $a\left(\frac{x}{\varepsilon}\right) \in C([0,\varepsilon^{-1}])$, again by the Weierstrass Theorem, it follows that there exist constants $B_1, B_2 > 0$ such that

$$\left|N_1\left(\frac{x}{\varepsilon}\right)\right| \leq B_1 , \ \left|a\left(\frac{x}{\varepsilon}\right)\right| \leq B_2 .$$

Thus, by Eq. (29) we have

$$\|F^\varepsilon\|^2_{L^2((0,1)\times(0,T))} \leq \varepsilon^2 A^2 T B^4 \int_0^{1/\varepsilon} dx = B^4 A^2 T \varepsilon,$$

where $B = \max\{B_1, B_2\}$, and so, it follows that

$$\|F^\varepsilon\|_{L^2((0,1)\times(0,T))} \leq \sqrt{T}\sqrt{\varepsilon} A B^2 . \tag{30}$$

Defining $c(T) = \sqrt{T}$ for some $T > 0$, we have from Eqs. (26) and (30) that

$$\|u^\varepsilon - u^{(1)}\|_{H_0^1((0,1)\times(0,T))} \leq \sqrt{\varepsilon} A B^2 \Longrightarrow \|u^\varepsilon - u^{(1)}\|_{H_0^1((0,1)\times(0,T))} = \mathcal{O}(\sqrt{\varepsilon}) . \tag{31}$$

Similarly, it is shown that

$$\|u^{(1)} - u_0\|_{H_0^1((0,1)\times(0,T))} = \mathcal{O}(\sqrt{\varepsilon}) . \tag{32}$$

Then, from Eqs. (31)and (32), it can be concluded

$$\begin{aligned}\|u^{\varepsilon}-u_0\|_{H_0^1((0,1)\times(0,T))} &= \|u^{\varepsilon}-u^{(1)}+u^{(1)}-u_0\|_{H_0^1((0,1)\times(0,T))} \\ &\leq \|u^{\varepsilon}-u^{(1)}\|_{H_0^1((0,1)\times(0,T))}+\|u^{(1)}-u_0\|_{H_0^1((0,1)\times(0,T))} \\ &= \mathcal{O}(\sqrt{\varepsilon})+\mathcal{O}(\sqrt{\varepsilon})=\mathcal{O}(\sqrt{\varepsilon})\ . \end{aligned} \tag{33}$$

Therefore, by Eq. (33), it follows that u_0 is an AE of u^{ε}.

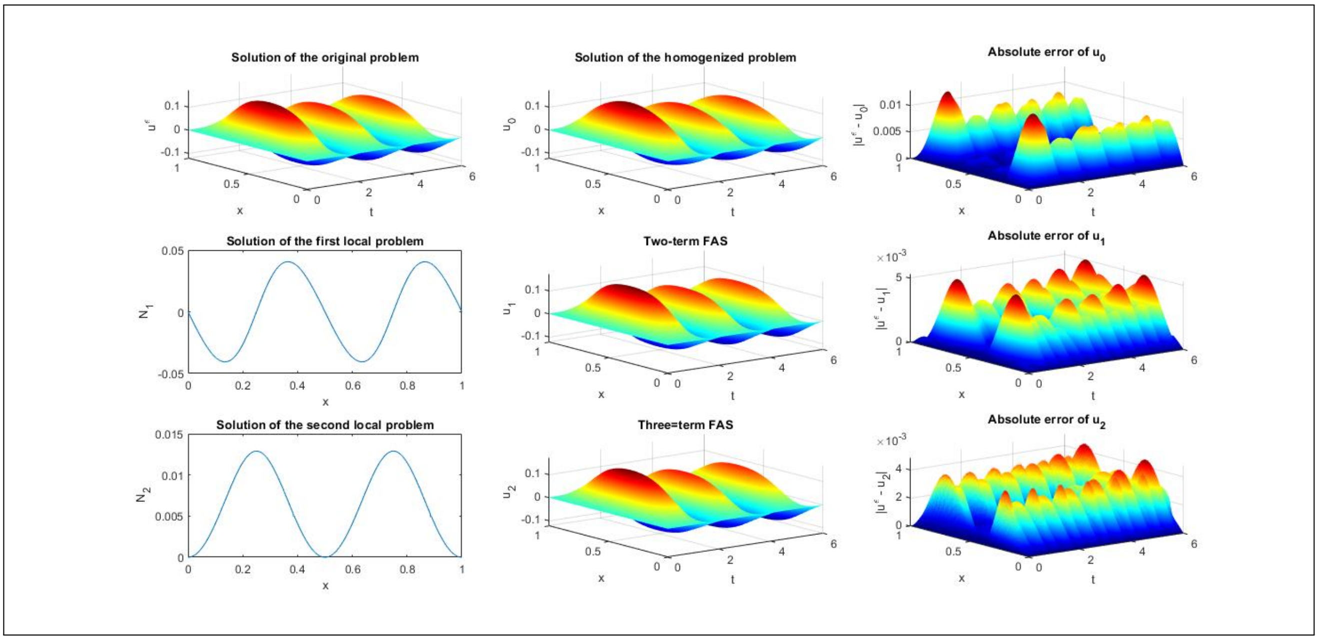
3 NUMERICAL EXAMPLE

To illustrate the effectiveness of the method, along with the formal proof, consider a problem for the wave equation in (2) with coefficient $a\left(\frac{x}{\varepsilon}\right)=1+0.25\cos\left(\frac{2\pi x}{\varepsilon}\right)$, body fource $f(x,t)=e^{-t}$ and subject to homogeneous boundary and initial conditions

$$\begin{cases} u^{\varepsilon}(0,t)=0,\ u^{\varepsilon}(1,t)=0 \\ u^{\varepsilon}(x,0)=0,\ \dfrac{\partial u^{\varepsilon}}{\partial t}(x,0)=0 \end{cases}.$$

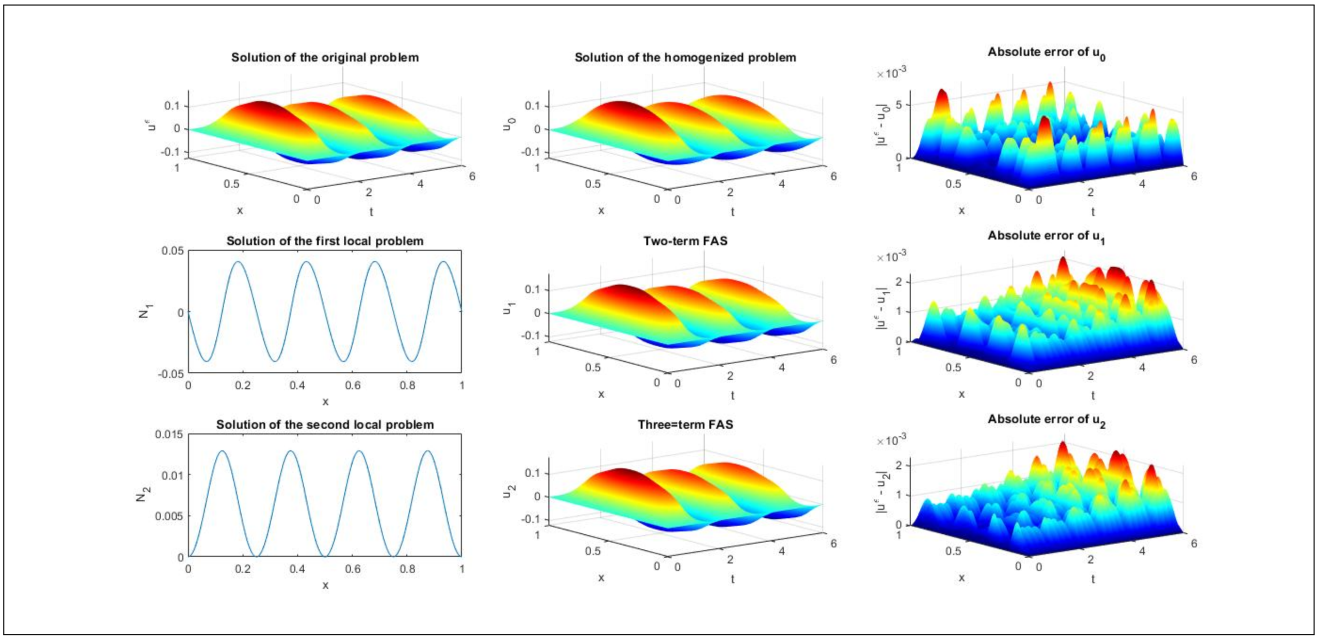
The figures below illustrate the surfaces for the solution of the original problem, the homogenized problem solution, the first and second order FAS, as well as presenting the absolute errors between the exact solution and the obtained approximations, given decreasing values of the small parameter. Additionally, graphs for the solutions of the local problems $N_1(y)$ and $N_2(y)$ are presented, respectively. It's possible to visualize the proximity between the solutions of the problems, meaning that as $\varepsilon \to 0^+$, the solution of the original problem converges to the solution of the homogenized problem.

Figure 1 – Behavior of the solutions of the original and homogenized problems, the FAS, the absolute error of each approximation, and the solutions of the local problems for $\varepsilon = 1/2$



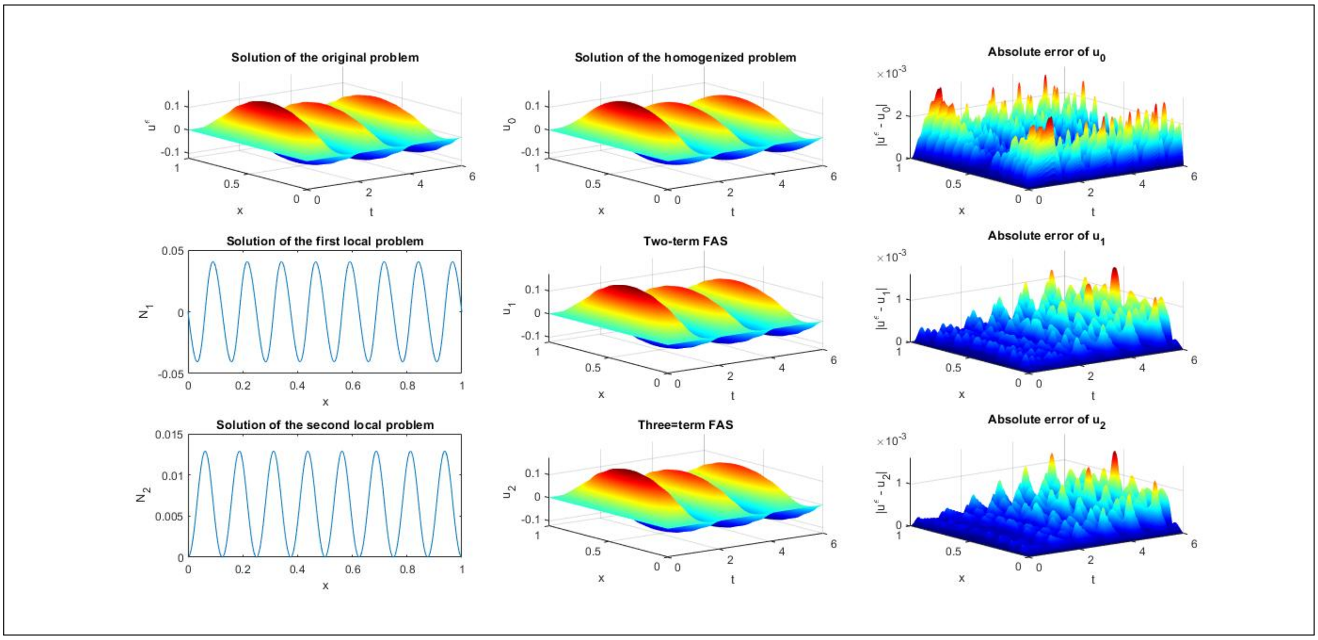
Source: the authors (2024)

Figure 2 – Behavior of the solutions of the original and homogenized problems, the FAS, the absolute error of each approximation, and the solutions of the local problems for $\varepsilon = 1/4$



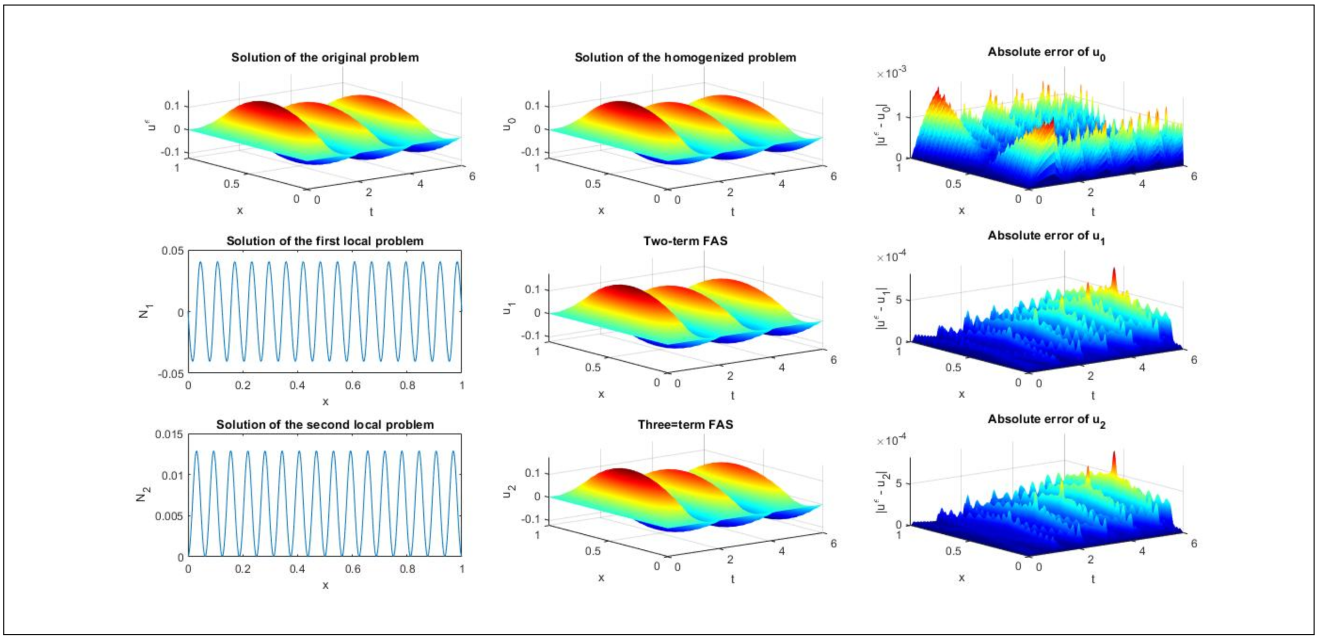
Source: the authors (2024)

Figure 3 – Behavior of the solutions of the original and homogenized problems, the FAS, the absolute error of each approximation, and the solutions of the local problems for $\varepsilon = 1/8$



Source: the authors (2024)

Figure 4 – Behavior of the solutions of the original and homogenized problems, the FAS, the absolute error of each approximation, and the solutions of the local problems for $\varepsilon = 1/16$



Source: the authors (2024)

4 CONCLUSIONS

The example presented illustrates that the AHM is a good alternative for dealing with problems whose structure is micro-heterogeneous, with coefficients oscillating rapidly. In fact, traditional approaches like the finite difference method would imply very fine meshes, given the purpose of capturing this rapidly oscillating behavior of the coefficients. This represents a high computational cost, which can interfere with the convergence of the solution. One advantage of using AHM is that, at least in structure, obtaining the solution of the homogenized problem is simpler to achieve, given that it presents constant coefficients. This arises from the respective problems presented in Eqs. (2) and (20).

Analyzing the presented graphs, it becomes evident that as the small parameter decreases, the second-order FAS becomes a better approximation of the original problem solution, as it provides more information about the problem's microstructure. In general, it shows that the three obtained approximations provide satisfactory results as $\varepsilon \to 0^+$.

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Author contributions

1 – Douglas Machado da Silva (Corresponding Author)
Student
https://orcid.org/0009-0006-5834-924X • doumach99@gmail.com
Contribution: Conceptualization; Data curation; Formal analysis; Investigation; Methodology; Software; Validation; Visualization; Writing-original draft; Writing – Review & Editing

2 – Leslie D. Pérez-Fernández
Mathematician, Professor
https://orcid.org/0000-0002-4452-264X • leslie.fernandez@ufpel.edu.br
Contribution: Conceptualization; Data curation; Formal analysis; Funding acquisition; Investigation; Methodology; Project administration; Resources; Software; Supervision; Validation; Visualization; Writing-original draft; Writing – Review & Editing

3 – Alexandre Molter
Mathematician, Professor
https://orcid.org/0000-0001-8562-6376 • Alexandre.molter@ufpel.edu.br
Contribution: Methodology;Resources; Software; Supervision; Writing – Review & Editing

4 – Julián Bravo-Castillero
Mathematician, Professor
https://orcid.org/0000-0002-7499-3821 • julian@mym.iimas.unam.mx
Contribution: Methodology; Project administration; Resources; Supervision; Writing – Review & Editing

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