CIÊNCIA e NATURA

ISSN 2179-460X
OPEN ACCESS

Ci. e Nat., Santa Maria, v. 46, n. spe. 1, 87225, 2024 • https://doi.org/10.5902/2179460X87225
Submitted: 03/22/2024 • Approved: 06/25/2024 • Published: 11/07/24

ERMAC e ENMC

The Salzer Summation and the numerical inversion of the Laplace Transform: performance analysis for oscillatory, exponential and logarithmic functions

A Soma de Salzer e a inversão numérica da Transformada de Laplace: análise de desempenho para funções oscilatórias, exponenciais e logarítmicas

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ABSTRACT

This article presents a study of the Salzer Summation, a technique for the numerical inversion of the Laplace Transform, applied to the inversion of five elementary functions with different behaviors: two oscillatory, two exponential and one logarithmic. Three of the functions studied have a variable parameter a (factor incorporated to investigate the efficiency of the method in dealing with functions of the same class). The algorithm's performance was analyzed for each value of M (number of terms in the sum) and parameter a chosen, through the Mean Absolute Error, graphical representation and execution times approximate. For the set of five functions presented (and for each a), the optimal value of M was determined. It was found that a does not significantly influence the execution time, unlike the parameter M, which directly interferes. Also, it was concluded that for oscillatory functions, the method presents convergence difficulties as the frequency increases.

Keywords: Laplace Transform; Laplace Inverse Transform; Gaver Functionals; Salzer Summation; Numerical methods

RESUMO

Neste artigo apresenta-se um estudo da Soma de Salzer, uma técnica para a inversão numérica da Transformada de Laplace, aplicada na inversão de cinco funções elementares com comportamentos diferentes: duas oscilatórias, duas exponenciais e uma logarítmica. Três das funções estudadas possuem um coeficiente variável a (fator incorporado a fim de investigar a eficiência do método em lidar com funções da mesma classe). O desempenho do algoritmo foi analisado, para cada valor de M

(número de termos no somatório) e parâmetro a escolhidos, através do Erro Absoluto Médio, da representação gráfica e dos tempos de execução aproximados. Para o conjunto de cinco funções apresentado (e para cada a), foi determinado o valor ótimo de M. Constatou-se que a não influencia de maneira relevante o tempo de execução, ao contrário do parâmetro M, que interfere diretamente. Também concluiu-se, que para funções oscilatórias, o método apresenta dificuldades de convergência na medida que a frequência aumenta.

Palavras-chave: Transformada de Laplace; Transformada Inversa de Laplace; Funcionais de Gaver; Soma de Salzer; Métodos numéricos

1 INTRODUCTION

In Science and Engineering, several mathematical models, such as the Heat Equation, Maxwell's Equations and the Equation of the Wave, are represented by Partial Differential Equations (PDEs). This type of equation can be solved analytically, using techniques such as the Variable Separation Method (Boyce and DiPrima, 2001), or numerically, through methods such as Finite Differences (Chapra and Canale, 2011). Another way to solve a PDE analytically is through the use of Integral Transforms, including the Laplace Transform.

Laplace Transform is a mathematical technique used to transform functions in time domains into functions in frequency (also called spectral) domains (Irwin, 2000). It has been widely applied in solving physical problems whose mathematical models are constituted by PDEs, among them, analysis of dynamic systems, such as electrical, mechanical and control (De Silva, 2023), as well as the study of filters and noise elimination (Buttkus, 2000).

Numerous problems require determining the solution in the time domain (Schiff, 2013). In these cases it is necessary to do the "way back", that is, transforming a function into the frequency domain into a function in the time domain. For this, it is needed to calculate the integral:

$$f(t) = \frac{1}{2\pi i} \int_C F(s) e^{st} ds \ , \tag{1}$$

where $F(s)$ is a function that depends on s, a complex variable, and C is a carefully chosen contour (Bellman et al., 1996). Solving the integral in (1) can become very laborious for highly complex functions, making it impossible to obtain an analytical solution, or even, depending on the type of problem addressed, even impossible

(Wang et al., 2017). For this reason, numerical methods for calculating the Inverse Laplace Transform become necessary. Several techniques have been developed to perform numerical shape inversion (Cohen, 2007), some of them are based on the deformation of the Bromwich contour and others apply the Fourier Series (Davies and Martin, 1979). Also noteworthy are the algorithms based on the Post-Widder Formula (Abate and Valkó, 2004); (Zakian, 1969), including the Gaver Functionals (Gaver, 1966).

Gaver Functionals are used to calculate the Inverse Laplace Transform numerically, however, their convergence is slow, requiring an acceleration scheme (Abate and Valkó, 2004). Among such schemes, Salzer Summation stands out, a linear acceleration method used in the Gaver-Stehfest algorithm (Stehfest, 1970).

Researchers, such as Abate and Valkó (2004), present five accelerators for Gaver Functionals and analyze their performance when applied to a set of elementary functions. In their studies, they evaluate that only the Wynn Rho Algorithm, among the schemes presented, is superior to the Salzer Summation. Dempsey and Duffy (2007) apply the Salzer Summation to solve a model on the acceleration and simultaneous radiative losses of electrons in the vicinity of relativistic shocks. Defreitas and Kane (2022) analyze the performance of five algorithms for the numerical inversion of the Laplace Transform: the expansion in Fourier Series, the Talbot Method with the contour deformation proposed by Valkó and Abate (2004), the Salzer acceleration scheme and the Stehfest accelerator for Gaver Functionals.

Within this context, and with the expectation of diversifying the applicability of the Salzer Summation in the numerical inversion of the Laplace Transform, this article presents a study on this technique when applied to a set of five elementary functions with different behaviors: two oscillatory, two exponential and a logarithmic one. Among the functions studied, three have a variable parameter a, a factor incorporated in order to investigate the efficiency of the method in dealing with functions of the same class. The study of the influence of this factor on the inversion process is important, as it determines the performance of the algorithm for ”extreme” behaviors, such as high frequencies or fast decays, cases presented in this work. Although the Salzer Summation is a simple and easy to implement technique, no references were found in the literature on the performance analysis of this technique for the Inverse Laplace Transform considering the incorporation of variable parameters.

To achieve the proposed objectives, this article is organized as follows: in section 2, the Gaver Functionals and the Salzer Summation are defined; in section 3, the set of elementary functions, the definition of the Mean Absolute Error, the results obtained and an analysis of the data are presented. Finally, in section 4, paper's conclusion and the proposal for continuity are discussed.

2 SALZER SUMMATION

There are many methods to solve the problem of numerical inversion of the Laplace Transform (Cohen, 2007). According to Abate and Valkó (2004), one of the most powerful and proven methods, the Gaver-Stehfest, involves using the so-called Gaver Functionals, which are given by

$$f_k(t) = (-1)^k \tau k \binom{2k}{k} \Delta^k F(k\tau) = k\tau \sum_{j=0}^{k} (-1)^j \binom{k}{j} F((k+j)\tau) \,, \tag{2}$$

where $\tau = \dfrac{\ln(2)}{t}$ and Δ is the forward difference operator, that is,

$$\Delta F(n\tau) = F((n+1)\tau) - F(n\tau). \tag{3}$$

The Gaver Functionals are derived from the Post-Widder Formula, where the derivatives are replaced by the difference operator (Adamek et al., 2017). The main advantages of using these functionals to numerically calculate the Inverse Laplace Transform are that the functionals depend only on F and do not have complex factors. However, it has the disadvantage of the logarithmic behavior of its convergence (Abate and Valkó, 2004). According to Valkó and Abate (2004), the Salzer Summation is a linear method to solve the problem of numerical inversion of the Laplace Transform, through the acceleration of the Gaver Functionals. Salzer Summation consists of fitting a Lagrange polynomial in $1/s$ to the transform of the function and inverting the polynomial term by term. The polynomial is fitted to the values of the transform at equal intervals along the positive real axis of the complex plane (Shirtliffe and Stephenson, 1961). Salzer's Summation expression for f is given by

$$f(t, M) \approx \sum_{k=1}^{M} W_k \; f_k(t), \tag{4}$$

where f_k is the Gaver Functional and the weights W_k are given by

$$W_k = (-1)^{k+M} \; \frac{k^M}{M!} \binom{M}{k}. \tag{5}$$

3 RESULTS

In this section, the performance of the Salzer Summation method in the numerical inversion of five elementary functions, presented in Table 1, where γ is Euler's constant, is analyzed. The results were obtained on a Dell Inspiron 15 700 Gaming notebook, with an Intel Core i5-7300HQ processor with 2.50GHz CPU, and 8 GB RAM, using the Octave software version 7.3.0. The parameter values chosen for the simulations were $a = \{1,\ 3,\ 10,\ 20\}$, $M \in \mathbb{N}$ from 1 to 14, $t \in [10^{-10},\ 1]$ with 101 points. It is worth highlighting that although results are shown up to $M = 14$, tests were carried out for M up to 20.

Table 1 – Laplace Transform for elementary functions

$F(s)$	$f(t)$
$F_1(s) = \dfrac{1}{(s+1)^2}$	$f_1(t) = te^{-t}$
$F_2(s) = \dfrac{a}{s^2 + a^2}$	$f_2(t) = \sin{(at)}$
$F_3(s) = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$	$f_3(t) = t\cos{(at)}$
$F_4(s) = -\dfrac{\ln{(s)}}{s}$	$f_4(t) = \ln{(t)} + \gamma$
$F_5(s) = \dfrac{1}{s+a}$	$f_5(t) = e^{-at}$

Source: the authors (2024)

The functions F_1 and F_5 were chosen because of its exponential behavior, F_2 and F_3 for its oscillatory behavior and F_4 for its logarithmic one. Three of the five functions also have a variable parameter a, which was incorporated to analyze the performance of the method for high frequencies, for oscillatory functions, and for high decays, for exponential functions.

The metric used to evaluate the method's performance was the Mean Absolute Error, which is defined here, as previously used in Calixto et al. (2022), as follows

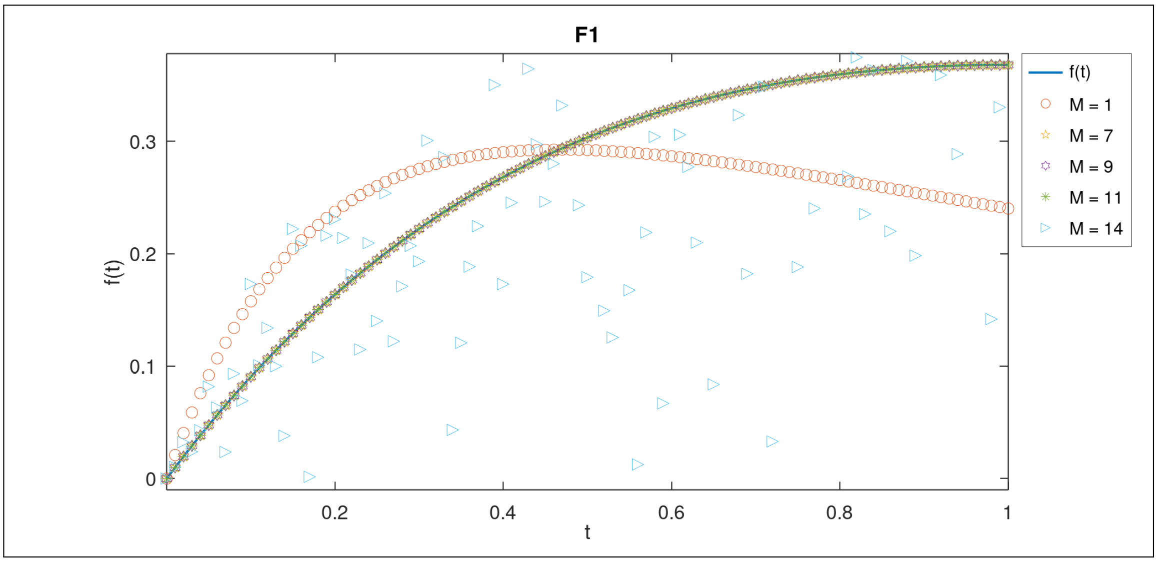
$$\overline{E}_{abs} = \frac{1}{N_t}\sum_{i=1}^{N_t} |\, f(t_i) - f^*(t_i)\,| \text{ , onde } t_i \in \mathbb{R}, \tag{6}$$

where f is the inverse of the Laplace Transform of the function F, f^* the numerical approximation of the inversion of F and $N_t \in \mathbb{N}$ the number of points t_i used in generating the profiles.

3.1 Function Inversion F_1

The Figure 1 and the Table 2 present the tests for the numerical inversion of the Laplace Transform for the function $F_1(s) = \dfrac{1}{(s+1)^2}$, as well as their approximate execution time.

Figure 1 – Results for the Salzer Summation in the inversion of F_1



Source: the authors (2024)

The optimal value for M in the inversion process of this function is 9. From values between $M = 7$ and $M = 10$, the Mean Absolute Errors vary between 10^{-8} to 10^{-7}.

3.2 Function Inversion F_2

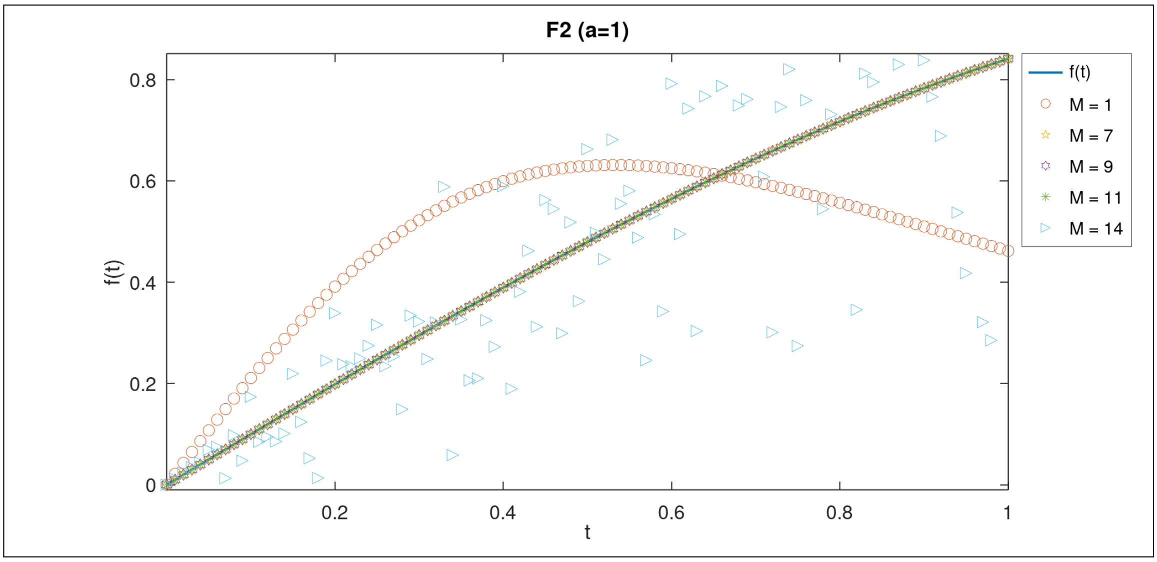
Figures 2 – 5 and Tables 3 and 4 show the tests for the numerical inversion of the Laplace Transform of the function $F_2(s) = \dfrac{a}{s^2 + a^2}$, as well as their approximate execution time.

Table 2 – Mean Absolute Errors and Approximate Execution Time in seconds, from F_1 to the Salzer Summation

M	$\overline{E}_{abs}$	Approximate Execution Time (s)
1	6.1818429×10^{-2}	0.1
2	1.3917656×10^{-2}	0.1
3	2.6828041×10^{-3}	0,2
4	4.3557818×10^{-4}	0.4
5	6.1282257×10^{-5}	0.5
6	7.2115268×10^{-6}	0.6
7	7.2732971×10^{-7}	0.7
8	6.3724613×10^{-8}	0.9
9	4.8901254×10^{-8}	1.1
10	9.2047346×10^{-7}	1.3
11	1.4753147×10^{-5}	1.5
12	2.8369893×10^{-4}	1.7
13	6.7047862×10^{-3}	2.0
14	1.2350562×10^{-1}	2.2

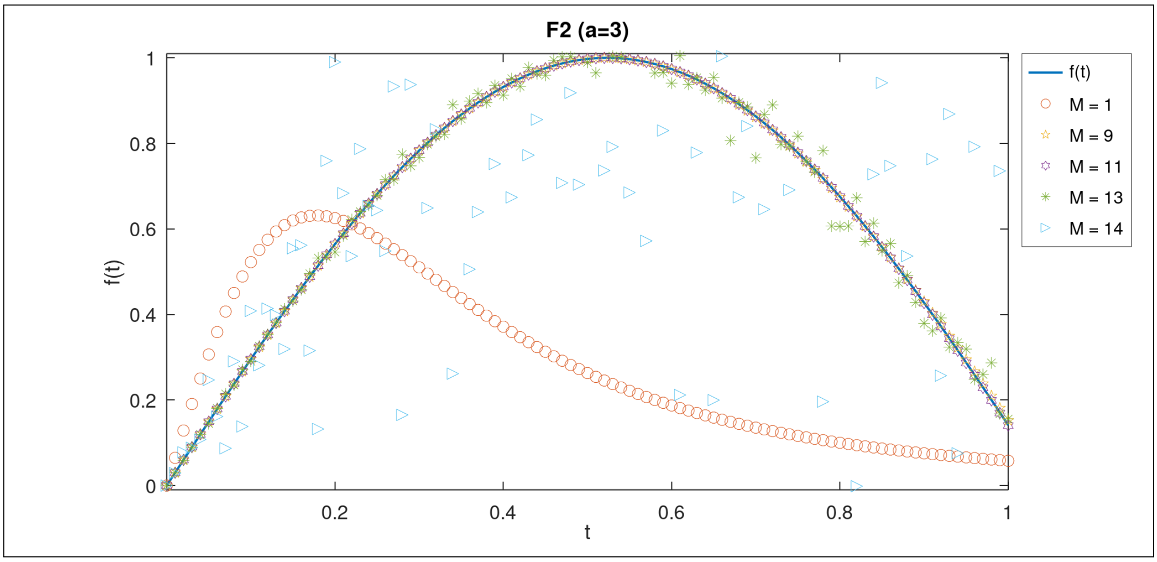
Source: the authors (2024)

Figure 2 – Results of the Salzer Summation in the inversion of F_2, with $a = 1$



Source: the authors (2024)

Figure 3 – Results of the Salzer Summation in the inversion of F_2, with $a = 3$



Source: the authors (2024)

Figure 4 – Results of the Salzer Summation in the inversion of F_2, with $a = 10$

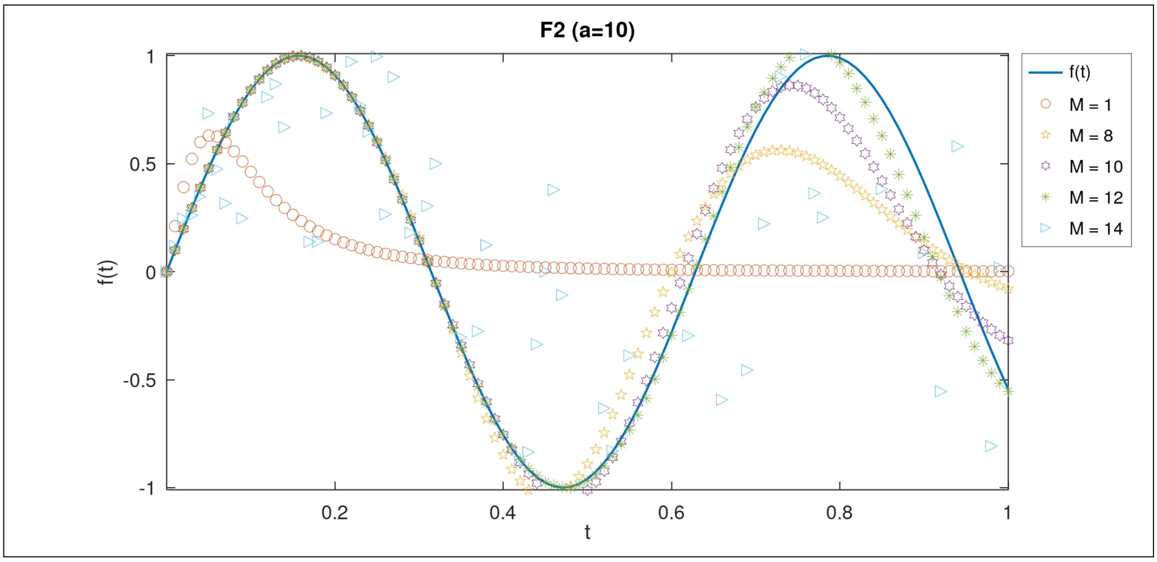


Figure 5 – Results of the Salzer Summation in the inversion of F_2, with $a = 20$

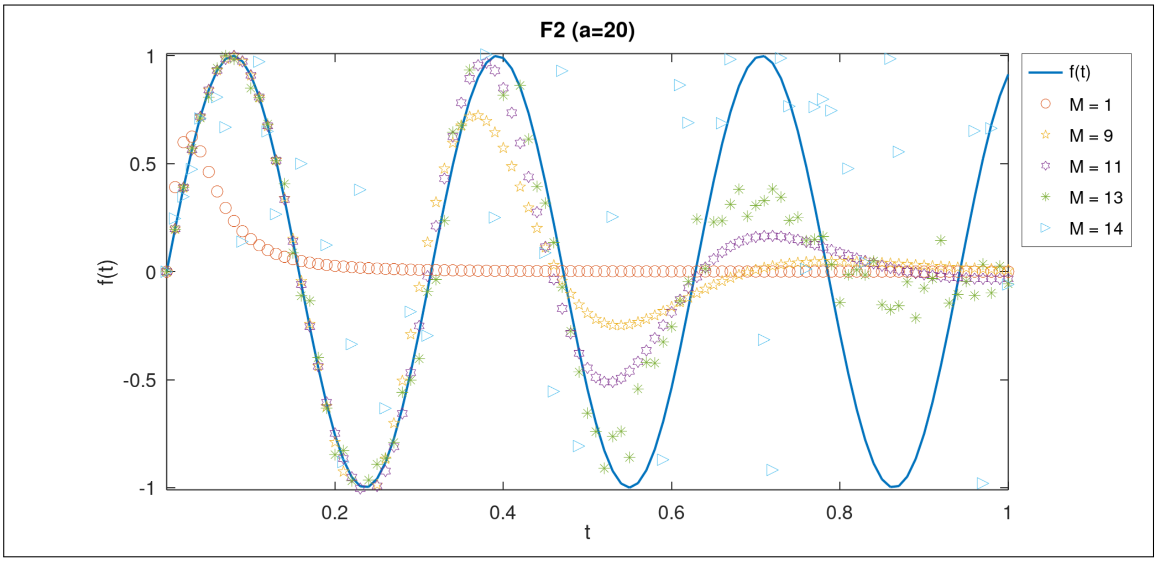


Table 3 – Mean Absolute Errors of F_2 for the Salzer Summation with $a = \{1,\ 3,\ 10,\ 20\}$

M	$a = 1$	$a = 3$	$a = 10$	$a = 20$
1	1.6172534×10^{-1}	4.3551024×10^{-1}	5.5096434×10^{-1}	5.9744862×10^{-1}
2	9.1215365×10^{-2}	2.8533410×10^{-1}	4.9425004×10^{-1}	5.6812802×10^{-1}
3	3.2546830×10^{-2}	1.8189732×10^{-1}	4.3336906×10^{-1}	5.3812970×10^{-1}
4	4.2150094×10^{-3}	1.0410129×10^{-1}	3.6971055×10^{-1}	5.0786677×10^{-1}
5	1.5046914×10^{-3}	4.8076472×10^{-2}	3.0722851×10^{-1}	4.7851421×10^{-1}
6	2.2872069×10^{-4}	1.7103246×10^{-2}	2.4877616×10^{-1}	4.4864223×10^{-1}
7	3.6048750×10^{-5}	6.8412845×10^{-3}	1.9703902×10^{-1}	4.1923322×10^{-1}
8	6.9730622×10^{-6}	3.8356610×10^{-3}	1.5200560×10^{-1}	3.8912729×10^{-1}
9	6.3555541×10^{-7}	1.1286617×10^{-3}	1.1483981×10^{-1}	3.5961890×10^{-1}
10	1.0828891×10^{-6}	2.6792442×10^{-4}	8.5117084×10^{-2}	3.2875391×10^{-1}
11	1.5575768×10^{-5}	1.6196967×10^{-4}	6.2515480×10^{-2}	2.9703053×10^{-1}
12	2.9543060×10^{-4}	8.2125302×10^{-4}	4.7574220×10^{-2}	2.6597426×10^{-1}
13	6.5414149×10^{-3}	1.9634249×10^{-2}	6.0908904×10^{-2}	2.5404514×10^{-1}
14	1.3530852×10^{-1}	3.7512292×10^{-1}	9.3379220×10^{-1}	1.2517640×10^{0}

The optimal values obtained for M varied between 9 and 13 and the Mean Absolute Errors were between 10^{-4} and 10^{-1} (except when $a = 1$). It was found that as the value of parameter a increases (consequent increase in the oscillation frequency) the value of the optimal M and the Mean Absolute Errors also increase.

Table 4 – Approximate execution times (in seconds) of the inversion of F_2 by the Salzer Summation with $a = \{1,\ 3,\ 10,\ 20\}$

M	$a = 1$	$a = 3$	$a = 10$	$a = 20$
1	0.1	0.1	0.1	0.1
2	0.1	0.1	0.1	0.1
3	0.2	0.2	0.2	0.2
4	0.3	0.3	0.3	0.3
5	0.5	0.5	0.5	0.5
6	0.6	0.6	0.6	0.6
7	0.7	0.7	0.7	0.7
8	0.9	0.9	0.9	0.9
9	1.1	1.1	1.1	1.1
10	1.3	1.3	1.3	1.3
11	1.5	1.5	1.5	1.5
12	1.7	1.7	1.7	1.7
13	2.0	2.0	2.0	2.1
14	2.4	2.4	2.2	2.2

3.3 Function Inversion F_3

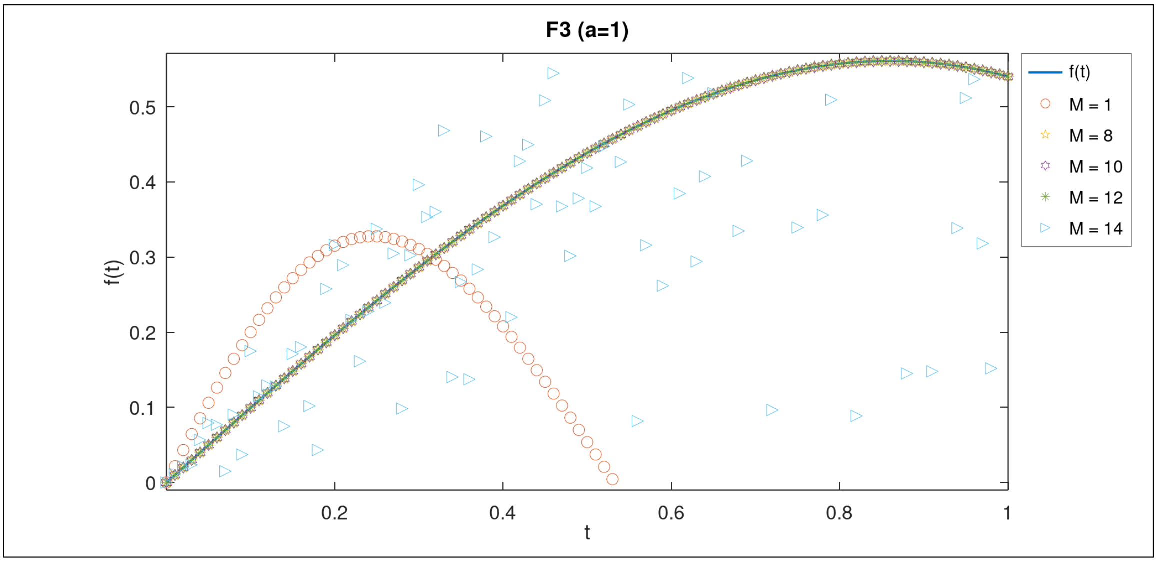
Figures 6 – 9 and Tables 5 and 6 present the tests for the numerical inversion of the Laplace Transform of the function $F_3(s) = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$, as well as its approximate execution time.

The optimal values for M varied between 10 and 14 for the different values of a, with Mean Absolute Errors between 10^{-4} and 10^{-1} (with the exception of $a = 1$). It was also observed that with the increase in the value of a there was also an increase in the value of the optimal M and the Mean Absolute Error.

3.4 Function Inversion F_4

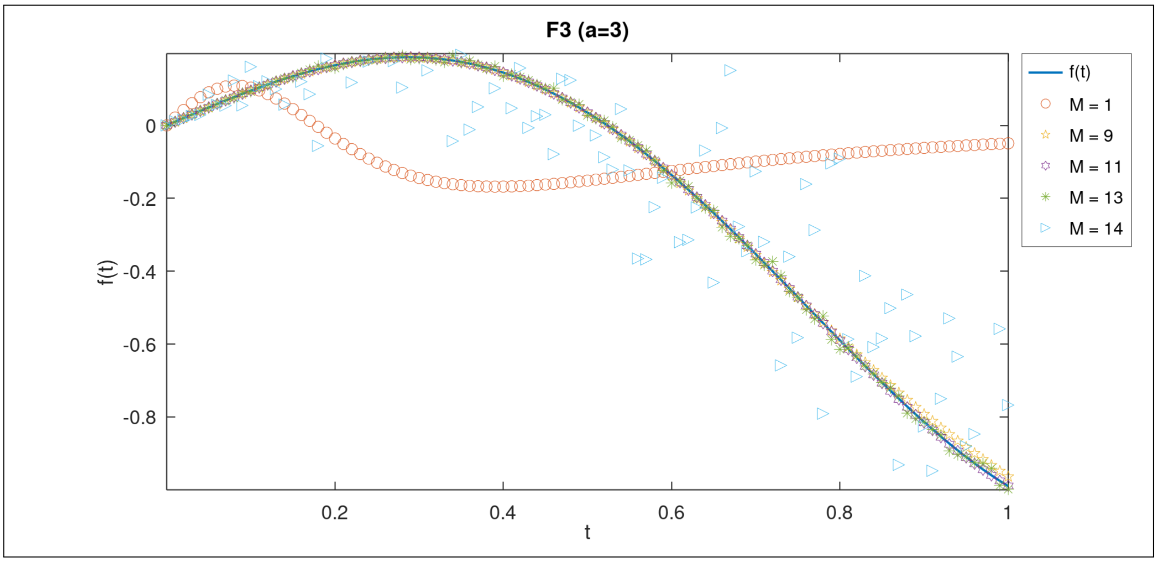
In Table 7 and Figure 10, there are tests for the numerical inversion of the Laplace Transform of the function $F_4(s) = -\dfrac{\ln(s)}{s}$, as well as their approximate execution time. The optimal value for M obtained for this function is 8. Between $M = 5$ and $M = 10$, the Mean Absolute Errors are between 10^{-5} and 10^{-7}.

Figure 6 – Results of the Salzer Summation in the inversion of F_3, with $a = 3$



Source: the authors (2024)

Figure 7 – Results of the Salzer Summation in the inversion of F_3, with $a = 3$



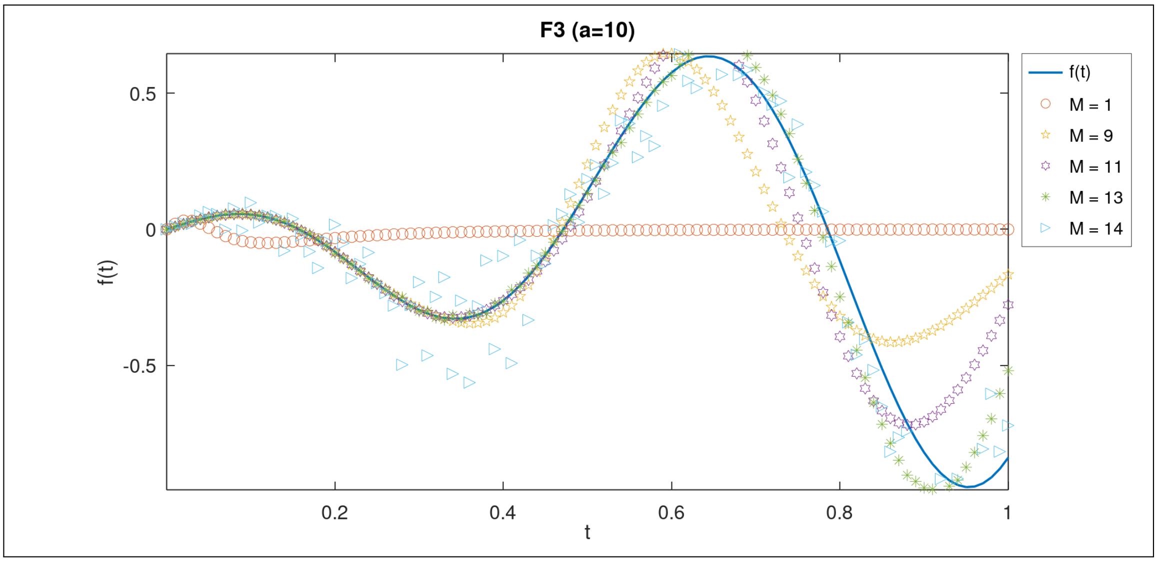
Source: the authors (2024)

3.5 Function Inversion F_5

Figures 11 – 14 and Tables 8 and 9 show the tests for the numerical inversion of the Laplace Transform of the function $F_5(s) = \dfrac{1}{s+a}$, as well as their approximate execution time.

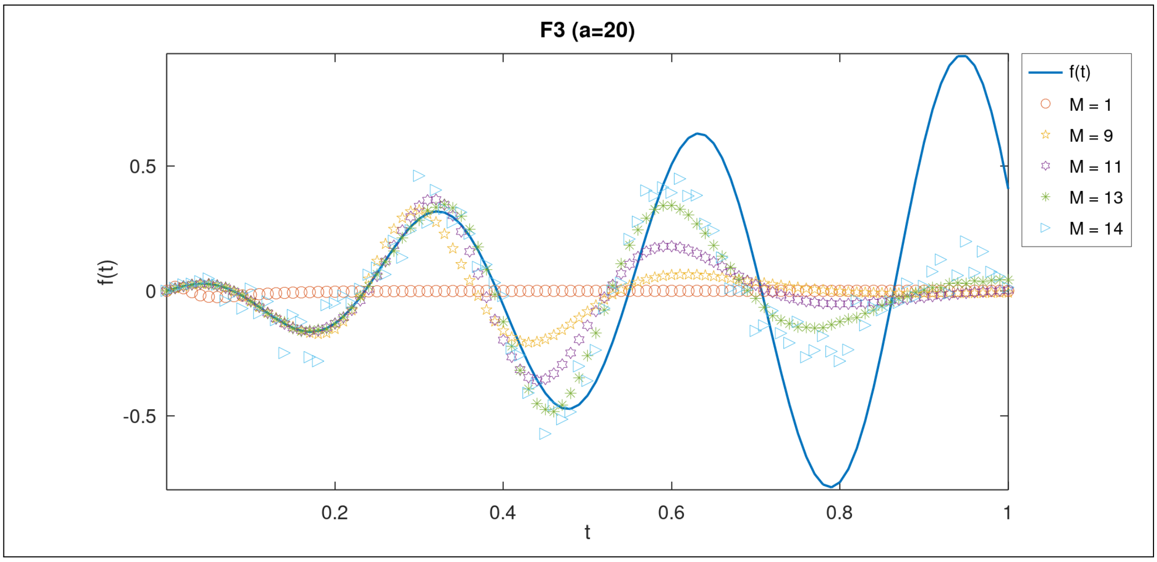
The optimal values for M varied between 8 and 9 for different values of a, resulting in profiles with Mean Absolute Errors between 10^{-8} and 10^{-6}. It was also verified that increasing the value of the parameter a implies in an increase in the Mean Absolute

Figure 8 – Results of the Salzer Summation in the inversion of F_3, with $a = 10$



Source: the authors (2024)

Figure 9 – Results of the Salzer Summation in the inversion of F_3, with $a = 20$



Source: the authors (2024)

Error.

3.6 General Analysis of Results

The results obtained validate the implemented algorithm, showing that the Salzer Summation can be used as an approximation for the Inverse Laplace Transform, generating results for 101 points in less than 3 seconds. The Table 10 shows the order of the Mean Absolute Errors for each optimal M in each parameter.

Table 5 – Mean Absolute Errors of the inversion of F_3 by the Salzer Summation with $a = \{1,\ 3,\ 10,\ 20\}$

M	$a = 1$	$a = 3$	$a = 10$	$a = 20$
1	4.6120171$\times10^{-1}$	3.0990295$\times10^{-1}$	3.3607875$\times10^{-1}$	3.2557774$\times10^{-1}$
2	1.7496329$\times10^{-1}$	2.6035701$\times10^{-1}$	3.2950155$\times10^{-1}$	3.2388590$\times10^{-1}$
3	8.6856647$\times10^{-2}$	1.8294133$\times10^{-1}$	3.1652144$\times10^{-1}$	3.2053194$\times10^{-1}$
4	3.0799947$\times10^{-2}$	1.1743898$\times10^{-1}$	2.9732729$\times10^{-1}$	3.1602780$\times10^{-1}$
5	4.1845241$\times10^{-3}$	9.0083395$\times10^{-2}$	2.7238610$\times10^{-1}$	3.1000520$\times10^{-1}$
6	1.8660826$\times10^{-3}$	5.4728659$\times10^{-2}$	2.4341184$\times10^{-1}$	3.0251396$\times10^{-1}$
7	1.6287479$\times10^{-4}$	2.0979460$\times10^{-2}$	2.1182811$\times10^{-1}$	2.9343018$\times10^{-1}$
8	6.1561883$\times10^{-5}$	7.3488411$\times10^{-3}$	1.7958042$\times10^{-1}$	2.8332236$\times10^{-1}$
9	4.5299109$\times10^{-6}$	4.4839409$\times10^{-3}$	1.4764983$\times10^{-1}$	2.7159759$\times10^{-1}$
10	1.8971494$\times10^{-6}$	1.2985203$\times10^{-3}$	1.1697789$\times10^{-1}$	2.5807977$\times10^{-1}$
11	1.5198850$\times10^{-5}$	3.4723285$\times10^{-4}$	8.8611543$\times10^{-2}$	2.4232606$\times10^{-1}$
12	2.7326679$\times10^{-4}$	3.6306540$\times10^{-4}$	6.3819837$\times10^{-2}$	2.2480385$\times10^{-1}$
13	6.4198169$\times10^{-3}$	6.1975035$\times10^{-3}$	4.4926194$\times10^{-2}$	2.0653632$\times10^{-1}$
14	1.3974708$\times10^{-1}$	1.3123718$\times10^{-1}$	9.0757997$\times10^{-2}$	2.0564736$\times10^{-1}$

Table 6 – Approximate execution time (in seconds) of the inversion of F_3 by the Salzer Summation with $a = \{1,\ 3,\ 10,\ 20\}$

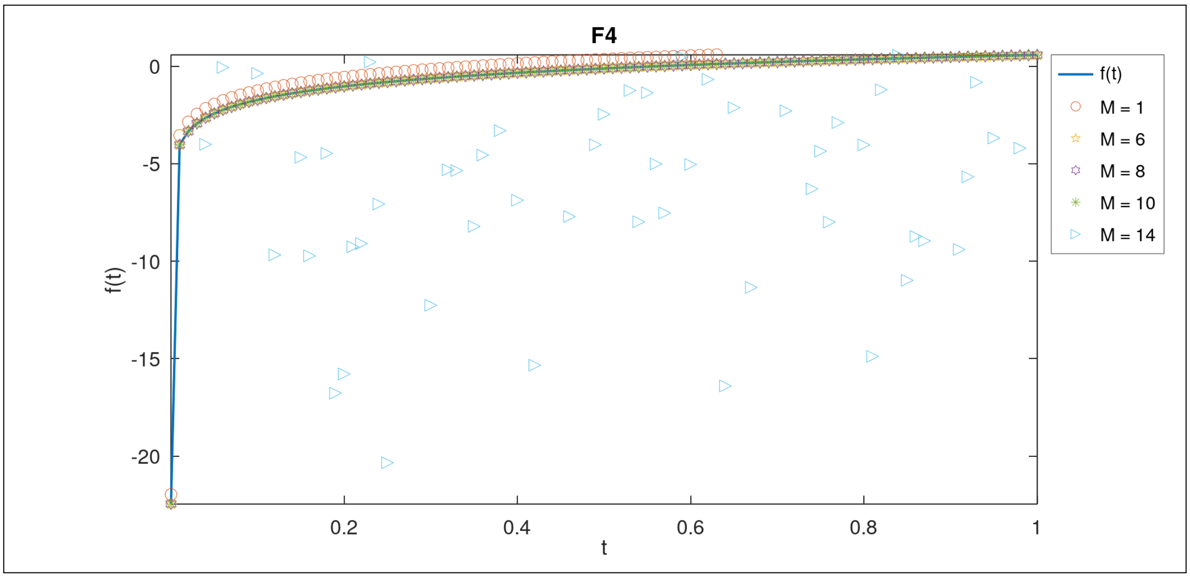
M	$a = 1$	$a = 3$	$a = 10$	$a = 20$
1	0.1	0.1	0.1	0.1
2	0.1	0.1	0.1	0.1
3	0.2	0.2	0.2	0.2
4	0.3	0.3	0.3	0.3
5	0.5	0.5	0.5	0.5
6	0.6	0.6	0.6	0.6
7	0.8	0.8	0.8	0.8
8	0.9	0.9	0.9	0.9
9	1.1	1.1	1.1	1.1
10	1.3	1.3	1.3	1.3
11	1.5	1.5	1.5	1.5
12	1.8	1.7	1.7	1.7
13	2.0	1.9	2.0	2.0
14	2.2	2.2	2.2	2.2

The Salzer Summation has a behavior of convergence to the optimal M, that is, starting with $M = 1$, the Mean Absolute Error decreases to a minimum value, after this value of M (which was called M optimal) the Mean Absolute Error grows indefinitely. For all functions tested using the established parameters, it was found that, for $M \geq 15$, the Mean Absolute Errors grow indefinitely, that is, the approximation becomes increasingly distant from the analytical result. This distancing occurs more intensely

Table 7 – Mean Absolute Errors and Approximate Execution Time (in seconds) of the inversion of F_4 by the Salzer Summation

M	$\overline{E}_{abs}$	Approximate Execution Time (s)
1	4.8244444×10^{-1}	0.1
2	3.8414360×10^{-2}	0.1
3	1.0787230×10^{-3}	0.2
4	1.8150750×10^{-4}	0.4
5	2.1756048×10^{-5}	0.5
6	3.9447587×10^{-7}	0.6
7	2.8912523×10^{-7}	0.8
8	1.1887447×10^{-7}	0.9
9	2.1241095×10^{-6}	1.1
10	4.5736268×10^{-5}	1.3
11	9.6545227×10^{-4}	1.5
12	1.9470304×10^{-2}	1.7
13	4.2628462×10^{-1}	2.0
14	9.7173020×10^{0}	2.2

Figure 10 – Results for the Salzer Summation in the inversion of F_4



Source: the authors (2024)

for functions with a trigonometric character (such as, for example, functions that have sines or cosines) when frequencies increase.

As for the influence of a and M on execution time, it was observed that a does not significantly interfere on execution time. However, the increase in the M parameter is directly proportional to the increase in execution time. It is also worth highlighting that there was no significant difference in time between the functions studied.

Table 8 – Mean Absolute Errors of the inversion of F_5 by the Salzer Summation with $a = \{1,\ 3,\ 10,\ 20\}$

M	$a = 1$	$a = 3$	$a = 10$	$a = 20$
1	1.4610357×10^{-1}	7.6432746×10^{-2}	3.2932203×10^{-2}	1.8360592×10^{-2}
2	2.2824012×10^{-2}	1.5333670×10^{-2}	1.0198904×10^{-2}	5.3628679×10^{-3}
3	2.8233266×10^{-3}	3.5571406×10^{-3}	2.7411819×10^{-3}	1.9094506×10^{-3}
4	3.0342846×10^{-4}	8.5811114×10^{-4}	9.4706238×10^{-4}	6.0816566×10^{-4}
5	3.0326244×10^{-5}	1.8963669×10^{-4}	3.0083795×10^{-4}	2.0954542×10^{-4}
6	2.8688662×10^{-6}	3.7053095×10^{-5}	7.5749026×10^{-5}	7.4645392×10^{-5}
7	2.2959137×10^{-7}	6.4182457×10^{-6}	2.2613279×10^{-5}	2.2344041×10^{-5}
8	3.4824247×10^{-8}	1.0058196×10^{-6}	7.3873202×10^{-6}	7.9838949×10^{-6}
9	6.3434324×10^{-7}	6.3531044×10^{-7}	2.3917821×10^{-6}	2.7447051×10^{-6}
10	1.2244549×10^{-5}	1.1277238×10^{-5}	1.0997507×10^{-5}	8.1281876×10^{-6}
11	2.3891850×10^{-4}	2.1111313×10^{-4}	2.0172039×10^{-4}	1.5872294×10^{-4}
12	4.6274079×10^{-3}	4.9586840×10^{-3}	4.1946608×10^{-3}	3.3405586×10^{-3}
13	9.9498671×10^{-2}	1.0605572×10^{-1}	8.0893070×10^{-2}	6.4701295×10^{-2}
14	1.3974708×10^{0}	2.1878647×10^{0}	1.5313168×10^{0}	1.3035556×10^{0}

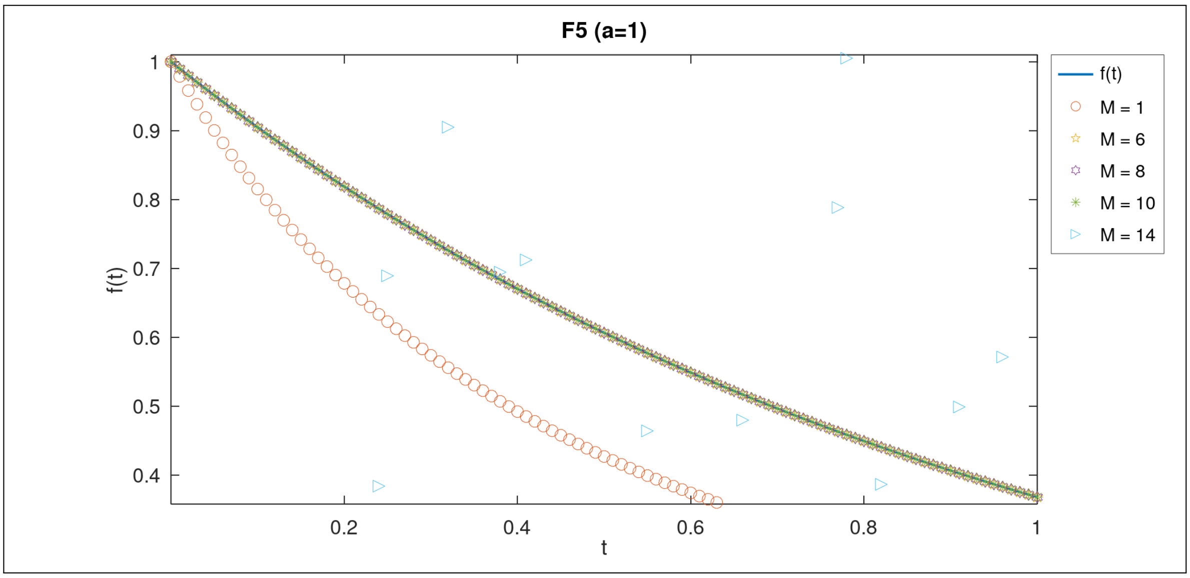
Table 9 – Approximate execution time (in seconds) of the inversion of F_5 by the Salzer Summation with $a = \{1,\ 3,\ 10,\ 20\}$

M	$a = 1$	$a = 3$	$a = 10$	$a = 20$
1	0.1	0.1	0.1	0.1
2	0.2	0.2	0.2	0.2
3	0.3	0.3	0.3	0.3
4	0.5	0.5	0.5	0.5
5	0.6	0.6	0.6	0.6
6	0.8	0.8	0.8	0.8
7	1.0	1.0	1.0	1.0
8	1.2	1.2	1.2	1.2
9	1.4	1.5	1.5	1.4
10	1.7	1.7	1.7	1.7
11	1.9	1.9	1.9	1.9
12	2.2	2.2	2.2	2.2
13	2.5	2.5	2.5	2.5
14	2.7	2.8	2.8	2.7

4 CONCLUSIONS

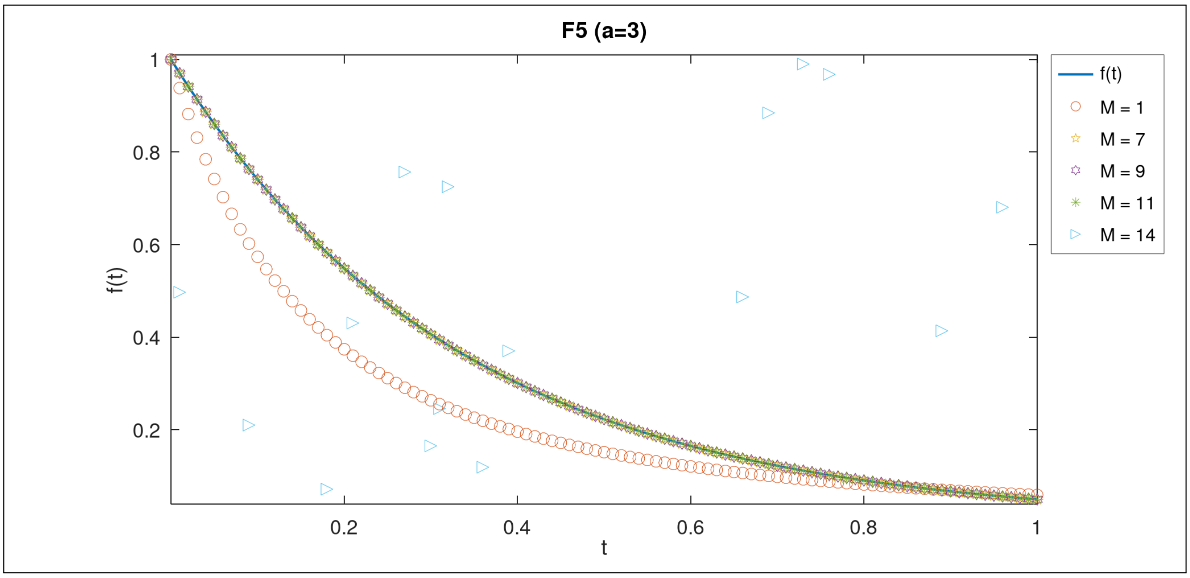
In this work, an analysis of the performance of the Salzer Summation in the inversion of oscillatory, exponential and logarithmic functions was presented. The influence of the parameters M and a on the performance of the method was verified, both through the Mean Absolute Error and graphical representation, as well as the

Figure 11 – Results of the Salzer Summation in the inversion of F_5, with $a = 1$



Source: the authors (2024)

Figure 12 – Results of the Salzer Summation in the inversion of F_5, with $a = 3$



Source: the authors (2024)

approximate execution time. The strategy chosen as the selection criterion for the optimal M allowed us not only to identify how many terms must be added in order to obtain good results, but also what the method's convergence limitations are. The value of a does not affect the execution time of the simulations, however it may imply the use of higher values of the optimal M to generate more accurate profiles. This occurs significantly in the inversion of high frequency oscillatory functions. However, F_5, which is an exponential function, is less sensitive to the variation of a compared to

Figure 13 – Results of the Salzer Summation in the inversion of F_5, with $a = 10$

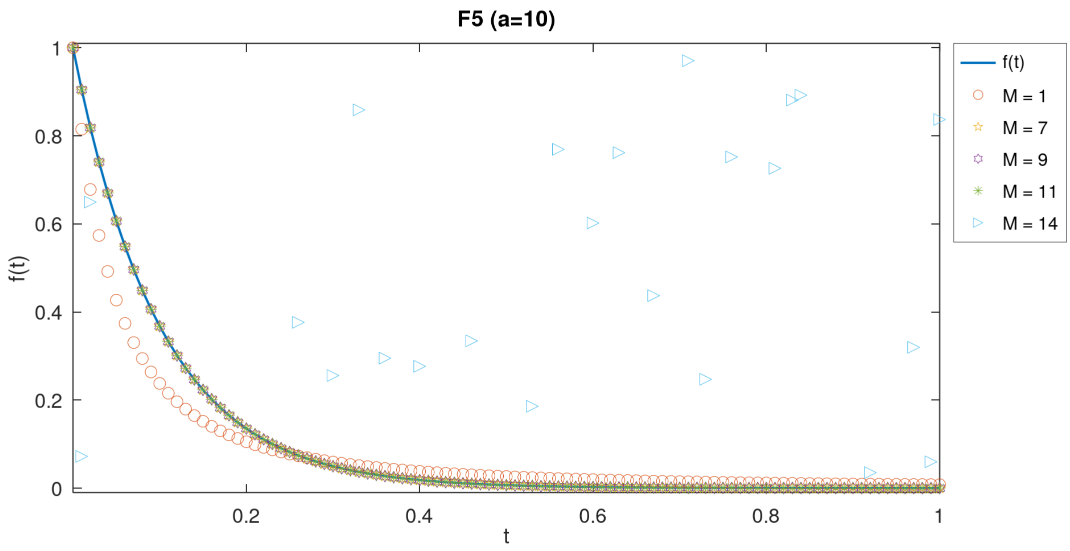


Figure 14 – Results of the Salzer Summation in the inversion of F_5, with $a = 20$

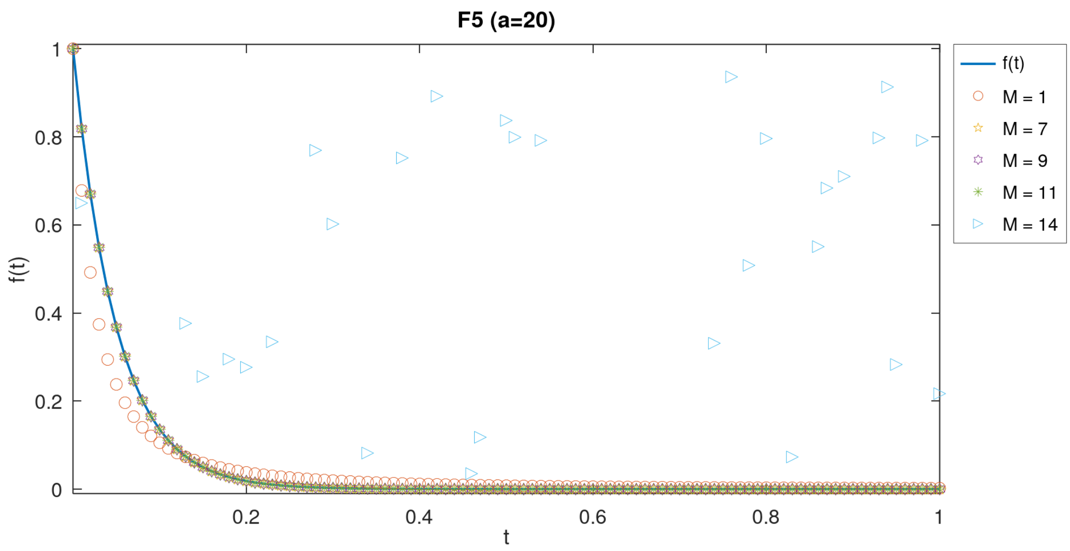


Table 10 – Comparisons between the optimal M and Order of Mean Absolute Error

	$a = 1$	$a = 3$	$a = 10$	$a = 20$
F_2	10^{-7} $[M = 9]$	10^{-4} $[M = 11]$	10^{-2} $[M = 12]$	10^{-1} $[M = 13]$
F_3	10^{-6} $[M = 10]$	10^{-4} $[M = 11]$	10^{-2} $[M = 13]$	10^{-1} $[M = 14]$
F_5	10^{-8} $[M = 8]$	10^{-7} $[M = 9]$	10^{-6} $[M = 9]$	10^{-6} $[M = 9]$
No Parameter				
F_1	10^{-8} $[M = 9]$			
F_4	10^{-7} $[M = 8]$			

oscillatory functions F_2 and F_3 .

Gaver Functionals are slowly converging numerical methods. Using the Salzer Summation as an accelerator allowed obtaining results comparable to the analytical solution at an extremely low computational cost (a few seconds). It should also be noted that as it does not depend on recursive schemes or usage of complex variables in the formulation, the implementation of the algorithms is simple and easy.

For future research, we intend to compare the performance of the Salzer Summation with other accelerators, such as Wynn Rho and Levin's u-transformation. It is also intended to apply such methods to the inversion of a set of functions with special characteristics, whose analytical inverse it is not easy to obtain (Freitas, 2022).

ACKNOWLEDGEMENTS

Thanks to CAPES for the financial support in carrying out this research.

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How to cite this article

Schmidt, R. A., Paz, M. C., Rodriguez, B. D. A., & Filho. J. F. P. (2024). The Salzer Summation and the numerical inversion of the Laplace Transform: performance analysis for oscillatory, exponential and logarithmic functions. *Ciência e Natura*, Santa Maria, v. 46, spe. 1, 87225. https://doi.org/10.5902/2179460X87225