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CIÊNCIA e NATURA

ISSN 2179-460X
OPEN ACCESS

Ci. e Nat., Santa Maria, v. 47, e84462, 2025• https://doi.org/10.5902/2179460X84462
Submitted: 07/17/23 • Approved: 10/04/23 • Published: 01/31/25

On second-order linear difference equations associated with hybrid sequences and generating functions

Sobre equações de diferenças lineares de ordem 2 associadas a sequências híbridas e funções geradoras

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ABSTRACT

This article presents explicit formulas for the class of homogeneous linear recurrence of order 2 with constant coefficients associated with hybrid sequences, determined through generating functions. Furthermore, the applications of the resolution method in each case are all displayed and the relations between Binet's formulas and the expressions obtained via generating functions are discussed. Illustrative examples are given to clarify the approach.

Keywords: Hybrid sequences; Linear homogeneous recurrences; Binet's formula; Generating functions

RESUMO

Este artigo apresenta fórmulas explícitas para a classe de recorrência linear homogênea de ordem 2 com coeficientes constantes associados à sequência híbrida, determinados através de funções geradoras. Além disso, as aplicações do método de resolução em cada caso são todos exibidos e as relações entre as fórmulas de Binet e as expressões obtidas via funções geradoras são discutidas. Exemplos ilustrativos são dados para esclarecer o abordado.

Palavras-chave: Sequências híbridas; Recidivas homogêneas lineares; Fórmula de Binet; Funções geradoras

1 INTRODUCTION

Complex, hyperbolic and dual numbers are two-dimensional systems known in the literature and researched in the last century for preliminary and physical application models. In Physics, when we interpret and demonstrate each space-time as being generated by the characteristic of the ideal of a hypercomplex ring, we can say that space-time is a structure generated by the algebra of hybrid numbers. This means that the hybrid numbers generate all space-times, then, through the study of hybrids, we can deduce in a purely algebraic way how each space-time is structured by numbers and how it evolved from coordinates.

The space where the events occur is a four-dimensional space called space-time, composed not only by the usual spatial directions, but also by a temporal direction. Furthermore, this space-time does not have a Euclidean structure, like that of three-dimensional physical space. In the world of mathematics, the four-dimensional space is given with the formalism of equations for can be understood and interpreted, but the need for four-dimensional thinking is still an obstacle, (see more in (Özdemir, 2018), and references therein).

Geometric algebras, also known as Clifford algebras, offer distinct advantages for formulating and analyzing physical models due to their associativity and the existence of an inverse element. Specifically, the geometric algebra of Euclidean space provides a comprehensive framework for classical areas of physics, including mechanics and electromagnetism, with a variety of benefits over traditional formulations.

When analyzing numerical sequences, it may be beneficial to consider the complex number system. By studying the difference equations that are relevant to this number set, we can uncover potential solutions. Additionally, we can apply established methods from the literature to better understand and analyze these solutions. In this context, this work consists of establishing a connection between solutions for difference equations and generating functions. More specifically, using the technique of ordinary generating functions, the solutions are obtained by solving homogeneous recurrence relations of order 2 with constant coefficients and initial values, without using resolution of Vandermonde systems.

In addition, this work focuses on various mathematical disciplines such as functions, polynomials, numerical sequences, progressions, linear systems, solving algebraic equations, and algebraic manipulations. The aim is to create a comprehensive resource for teachers that will enable them to use algebraic knowledge to solve homogeneous linear recurrences of order 2 with constant coefficients, using explicit formulas. All of this is applicable to Basic Education. On the other hand, we show clear examples of counting problems, which can serve as material for the teacher to apply directly in the classroom. In addition to the resolution of the associated recurrence, we emphasize here the case of simple roots that is more common in the literature, thus bringing a possible didactic sequence that the teacher can use in the classroom.

The content of this article is as follows. In Section 2, we give a brief overview of generating function as a power series and provide the generating function of homogeneous second-order recurrence with constant coefficients associated with hybrid sequence, with initial conditions. In Section 3, we use the generating function to establish the solution of homogeneous second-order recurrence with constant coefficients associated with hybrid sequence, with initial conditions. Section 4 concerns the connection between the analytic form (Binet's formula) given in terms of roots of characteristic polynomial and the results given in Section 3. Finally, Section 5 is dedicated to final considerations, with a discussion on the scope of our approach both in terms of teacher training and in terms of didactic approach.

2 GENERATING FUNCTION AND LINEAR DIFFERENCE EQUATION OF ORDER 2 ASSOCIATED WITH HYBRID SEQUENCE

2.1 Generating function and power series

Generating functions are power series where the coefficients give us information about a sequence $(a_n)_{n\in\mathbb{N}}$ and the exponent of the variable in the series quantifies some property we are interested in about this sequence. If we associate such powers of the variable x by adding them together, the coefficient of x^n will be the term of the sequence at position n that presented the solution in the problem, (see details in (Lima, 2009)). Follows the definition,

Definition 2.1. *Power series or formal series are infinite series of the form*

$$\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

where each term is a constant multiplied by a power of x.

In the combinatorial point of view, it is not necessary to observe the convergence aspect of the power series. However, if the variable x takes on a specific numeric value, the series becomes a constant sequence of terms that either converges or diverges. A power series may converge for some values of x and diverge for others. It can be shown that to each power series there corresponds a symmetric interval $-L < x < L$, inside which the series converges and outside which it diverges. At the edge points, $x = -L$ and $x = L$, it can either converge or diverge. The number L is called the *radius of convergence*, and the set of all numbers for which the series converges is called the *interval of convergence*. This interval can be infinite if the series is convergent for all values of x in the set of real numbers (see (Lima, 2009) for more details).

Example 2.1. *The geometric series* $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ *has convergence ratio* $\mathrm{L} = 1$ *and is derivable in* $(-1{,}1)$. *Then,*

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \sum_{n=1}^{\infty} nx^{n-1}.$$

2.2 Linear Difference Equation of order 2 associated with hybrid sequence

Recently, many studies have been devoted to hybrid numbers whose components are taken from special integer sequences, such as Fibonacci, Lucas, Pell and Jacobsthal, (see (Özdemir, 2018), (Szynal-Liana, 2018), (Szynal-Liana & Wlock, 2019a), (Catarino, 2019)) and references therein). Hybrid numbers have an algebraic and geometric structure, but especially when they are elements of recurrence sequences, such as Horadam and Leonardo numbers, they define complex generalizations of generalized Fibonacci numbers. This new non-commutative numbering system, the hybrid numbers, was introduced by Ozdemir (Özdemir, 2018) as a generalization of complex numbers, dual numbers, and hyperbolic numbers, that are defined as the set

$$\mathbb{H} = \{a + bi + c\epsilon + dh, \mid i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i\}.$$

In particular, Szynal-Liana (Szynal-Liana, 2018) introduced the hybrid numbers of Horadam and several properties of special types of hybrid numbers were explored, (Szynal-Liana & Wlock, 2019a), (Szynal-Liana & Wlock, 2019b), (Szynal-Liana & Wlock, 2018). Furthermore, Morales (Morales, 2018) worked on a generalization of the hybrid numbers of $(p, q)-$Fibonacci and $(p, q)-$Lucas obtaining new identities between them. Another class explored was the $k-$Pell hybrid numbers, investigated by Paula Catarino (Catarino, 2019).

On the other hand, the sequences defining the generalized Fibonacci numbers, the generalized Pell numbers or the Jacobsthal numbers, are special cases of the sequences defined by linear recurrence relations.

Consider the sequence $(V_n)_{n\geq 0}$ of order 2, defined by

$$V_n + pV_{n-1} + qV_{n-2} = 0 \qquad \text{for} \qquad n \geq 2, \tag{1}$$

where $V_0 = a_0$, $V_1 = a_1$, and p, q are fixed real or complex numbers.

The hybrid number $\mathbb{H}V_n$ is defined by the relation

$$\mathbb{H}V_n = V_n + V_{n+1}i + V_{n+2}\epsilon + V_{n+3}h, \tag{2}$$

where i,ϵ and h are hybrid units. Replacing the relation (1) in (2), we get the following identities,

$$p\mathbb{H}V_{n-1} = pV_{n-1} + pV_n i + pV_{n+1}\epsilon + pV_{n+2}h \tag{3}$$

$$q\mathbb{H}V_{n-2} = qV_{n-2} + qV_{n-1}i + qV_n\epsilon + qV_{n+1}h \tag{4}$$

By Expressions (3) and (4) we obtain the following recurrence relation,

$$\mathbb{H}V_n + p\mathbb{H}V_{n-1} + q\mathbb{H}V_{n-2} = 0, \tag{5}$$

with the initial conditions,

$$\mathbb{H}V_0 = a_0 + a_1 i - (pa_1 + qa_0)\epsilon + [(p^2 - q)a_1 + pqa_0]h$$

$$\mathbb{H}V_1 = a_1 - (pa_1 + qa_0)i + [(p^2 - q)a_1 + pqa_0)]\epsilon + [(-p^3 + 2pq)a_1 + (q^2 - p^2 q)a_0]h \tag{6}$$

The Expression (5) shows that the recurrence relation associated with the hybrid numbers is a homogeneous linear recurrence of order 2 with initial conditions. In addition, since the coefficients of the recurrence relation and the hybrid recurrence relation associated are the same, the characteristic polynomial and their roots are the same. Under these observations, the approach to solving a homogeneous linear recurrence of order 2 can be applied to solve the hybrid recurrence relation associated.

2.3 The generating function

In this section we will apply the generating function technique to find an explicit formula for a second-order homogeneous linear recurrence with constant coefficients associated with hybrid sequence. Consider the linear recurrence of order 2 given by Equation (5) with the initial conditions (6). Using the concept of power series, we define

$$\mathbb{H}f(x) = \sum_{n=0}^{\infty} \mathbb{H}V_n x^n.$$

By multiplication of x^n in both sides of equality (5), we obtain,

$$\begin{aligned}
\mathbb{H}V_n x^n &= -p\mathbb{H}V_{n-1}x^n - q\mathbb{H}V_{n-2}x^n, \\
\mathbb{H}V_n x^n &= -px\mathbb{H}V_{n-1}x^{n-1} - qx^2\mathbb{H}V_{n-2}x^{n-2}, \\
\sum_{n=2}^{\infty} \mathbb{H}V_n x^n &= -px\sum_{n=2}^{\infty} \mathbb{H}V_{n-1}x^{n-1} - qx^2 \sum_{n=2}^{\infty} \mathbb{H}V_{n-2}x^{n-2}, \\
-\mathbb{H}V_0 - \mathbb{H}V_1 x + \sum_{n=0}^{\infty} \mathbb{H}V_n x^n &= px\mathbb{H}V_0 - px\sum_{n=0}^{\infty} \mathbb{H}V_n x^n - qx^2 \sum_{n=0}^{\infty} \mathbb{H}V_n x^n.
\end{aligned}$$

Then,

$$(1 + px + qx^2)\sum_{n=0}^{\infty} \mathbb{H}V_n x^n = \mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x,$$

$$\mathbb{H}f(x) = \frac{\mathbb{H}V_0 + (\mathbb{H}V_1 + p\mathbb{H}V_0)x}{1 + px + qx^2}. \tag{7}$$

Expression (7) is the generating function to the linear recurrence $\mathbb{H}V_n + p\mathbb{H}V_{n-1} + q\mathbb{H}V_{n-2} = 0$. In fact, this result is a particular case of a result published in the article (Craveiro et al., 2022), that established the explicit formula for the generating functions for the homogeneous linear recurrence relation of order 2, namely,

Theorem 1 (Theorem 3.2, (Craveiro et al., 2022))**.** *Given a linear homogeneous recurrence with constant coefficients* p, q*, where* $q \neq 0,$ *initial conditions* a_0 *and* $a_1,$ *and equation* $a_{n+2} + pa_{n+1} + qa_n = 0, n \geq 0,$ *then the generating function of sequence* $(a_n)_{n\geq 0}$ *is equal to*

$$f(x) = \frac{a_0 + a_1x + pxa_0}{1 + px + qx^2}.$$

The direct application of Theorem 1 for Equation (5) with the initial conditions (6) give us the same result via generating function.

Proposition 2.1. *Consider the homogeneous linear recurrence with constant coefficients* p, q*, where* $q \neq 0,$ *associated with the hybrid numbers* $(\mathbb{H}V_n)_{n\geq 0}$ *with initial conditions* $\mathbb{H}V_0,$ $\mathbb{H}V_1,$ (6)*, and equation* $\mathbb{H}V_{n+2} + p\mathbb{H}V_{n+1} + q\mathbb{H}V_n = 0, n \geq 0,$ (5)*. Then, the generating function of sequence* $(\mathbb{H}V_n)_{n\geq 0}$ *is equal*

$$\mathbb{H}f(x) = \frac{\mathbb{H}V_0 + (\mathbb{H}V_1 + p\mathbb{H}V_0)x}{1 + px + qx^2}.$$

Note that the Expression (7) is a rational function. With a new perspective, we focus in analyse the denominator of Expression (7) and provide an explicit formula for the coefficient x^n in expansion of $\mathbb{H}f(x)$. By considering p and q real numbers, since $1 + px + qx^2$ is a second degree polynomial, there are three possibilities for the discriminant $\Delta = p^2 - 4q$, $\Delta > 0$,$\Delta < 0$ or $\Delta = 0$. The next section is devoted to study each case of discriminant.

3 EXPLICIT FORMULAS

3.1 The case $\Delta > 0$ of polynomial $1 + px + qx^2$

Let $\Delta = p^2 - 4q > 0$ be the discriminant of polynomial $s(x) = 1 + px + qx^2$. Since $\Delta > 0$, then $s(x)$ have simple roots $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$qx^2 + px + 1 = q(x - \lambda_1)(x - \lambda_2) \tag{8}$$

where, $\lambda_1 = \dfrac{-p + \sqrt{p^2 - 4q}}{2q}$ and $\lambda_2 = \dfrac{-p - \sqrt{p^2 - 4q}}{2q}, q \neq 0.$

Since λ_1,$\lambda_2 \neq 0,$ then the Expressions (7) and (8) allow us to obtain

$$\mathbb{H}f(x) = \frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{1 + px + qx^2} = \left(\frac{1}{q}\right)\left[\frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{(x - \lambda_1)(x - \lambda_2)}\right]. \tag{9}$$

Decomposing (9) into partial fractions (more details in (Lima, 2009)) we obtain,

$$\begin{aligned}\frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{(x - \lambda_1)(x - \lambda_2)} &= \left(\frac{1}{\lambda_1 - \lambda_2}\right)\left(\frac{\mathbb{H}V_0 + \mathbb{H}V_1 \lambda_1 + p\lambda_1\mathbb{H}V_0}{x - \lambda_1}\right) \\ &+ \left(\frac{1}{\lambda_2 - \lambda_1}\right)\left(\frac{\mathbb{H}V_0 + \mathbb{H}V_1 \lambda_2 + p\lambda_2\mathbb{H}V_0}{x - \lambda_2}\right).\end{aligned} \tag{10}$$

By replacing Expression (10) in (9) follows the result,

$$\begin{aligned}\mathbb{H}f(x) = \frac{1}{q}&\left[\frac{\mathbb{H}V_0 + \mathbb{H}V_1 \lambda_1 + p\lambda_1\mathbb{H}V_0}{\lambda_1 - \lambda_2}\left(\frac{1}{1 - \left(\frac{x}{\lambda_1}\right)}\right)\left(\frac{-1}{\lambda_1}\right)\right. \\ &\left.+ \frac{\mathbb{H}V_0 + \mathbb{H}V_1 \lambda_2 + p\lambda_2\mathbb{H}V_0}{\lambda_2 - \lambda_1}\left(\frac{1}{1 - \left(\frac{x}{\lambda_2}\right)}\right)\left(\frac{-1}{\lambda_2}\right)\right].\end{aligned} \tag{11}$$

By considering $\left|\dfrac{x}{\lambda_1}\right| < 1$ and $\left|\dfrac{x}{\lambda_2}\right| < 1,$ the geometric series $\dfrac{1}{1 - \left(\frac{x}{\lambda_1}\right)}$ and $\dfrac{1}{1 - \left(\frac{x}{\lambda_2}\right)}$ converge. Then, we obtain,

$$\mathbb{H}f(x) = \frac{-1}{q}\left[\sum_{n=0}^{\infty}\left(\frac{\mathbb{H}V_0 + \mathbb{H}V_1 \lambda_1 + p\lambda_1\mathbb{H}V_0}{\lambda_1(\lambda_1 - \lambda_2)}\right)\left(\frac{x}{\lambda_1}\right)^n + \left(\frac{\mathbb{H}V_0 + \mathbb{H}V_1 \lambda_2 + p\lambda_2\mathbb{H}V_0}{\lambda_2(\lambda_2 - \lambda_1)}\right)\left(\frac{x}{\lambda_2}\right)^n\right],$$

where the coefficient of x^n is given by

$$\frac{-1}{q}\left[\frac{\mathbb{H}V_0+\mathbb{H}V_1\lambda_1+p\mathbb{H}V_0\lambda_1}{(\lambda_1-\lambda_2)\lambda_1^{n+1}}+\frac{\mathbb{H}V_0+\mathbb{H}V_1\lambda_2+p\mathbb{H}V_0\lambda_2}{(\lambda_2-\lambda_1)\lambda_2^{n+1}}\right]. \tag{12}$$

Under the previous discussion, follows the result,

Proposition 3.1. *Consider the homogeneous linear recurrence with constant coefficients p, q, where $q \neq 0,$ associated with the hybrid numbers $(\mathbb{H}V_n)_{n\geq 0}$ with initial conditions $\mathbb{H}V_0$ and $\mathbb{H}V_1,$* (6)*, and equation $\mathbb{H}V_{n+2}+p\mathbb{H}V_{n+1}+q\mathbb{H}V_n=0, n\geq 0,$* (5)*. If the associated polynomial $s(x)=1+px+qx^2$ has discriminant $\Delta=p^2-4q>0,$ then the explicit formula for $\mathbb{H}V_n$ is given by,*

$$\mathbb{H}V_n=\frac{-1}{q}\left[\frac{\mathbb{H}V_0+\mathbb{H}V_1\lambda_1+p\mathbb{H}V_0\lambda_1}{(\lambda_1-\lambda_2)\lambda_1^{n+1}}+\frac{\mathbb{H}V_0+\mathbb{H}V_1\lambda_2+p\mathbb{H}V_0\lambda_2}{(\lambda_2-\lambda_1)\lambda_2^{n+1}}\right].$$

where λ_1 and λ_2 are simple roots of $s(x)$.

The Proposition 3.1 gives us the explicit formula for $\mathbb{H}V_n$ in the case $\boldsymbol{\Delta>0}$.

Example 3.1. *Consider the usual Fibonacci sequence given by $F_n=F_{n-1}+F_{n-2}, n\geq 2,$ and initial conditions $F_0=0, F_1=1.$ The hybrid sequence associated with Fibonacci numbers is given by,*

$$\mathbb{H}F_{n+2}-\mathbb{H}F_{n+1}-\mathbb{H}F_n=0 \tag{13}$$

for $n\geq 0,$ with initial conditions $\mathbb{H}F_0=i+\epsilon+2h, \mathbb{H}F_1=1+i+2\epsilon+3h.$

The polynomial associated $s(x)=1-x-x^2$ has roots $\lambda_1=\dfrac{-1-\sqrt{5}}{2}$ and $\lambda_2=\dfrac{-1+\sqrt{5}}{2}$. Then, by direct application of Proposition 3.1, is derived the explicit formula,

$$\mathbb{H}F_n=\left[\frac{(i+\epsilon+2h)+(1+i+2\epsilon+3h)\left(\frac{-1-\sqrt{5}}{2}\right)-(i+\epsilon+2h)\left(\frac{-1-\sqrt{5}}{2}\right)}{-\sqrt{5}\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}}\right]$$
$$+\left[\frac{(i+\epsilon+2h)+(1+i+2\epsilon+3h)\left(\frac{-1+\sqrt{5}}{2}\right)-(i+\epsilon+2h)\left(\frac{-1+\sqrt{5}}{2}\right)}{\sqrt{5}\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}\right].$$

3.2 The case $\Delta = 0$ of polynomial $1 + px + qx^2$

Recall the denominator of $\mathbb{H}V_n$ in Theorem 2.1, the polynomial $s(x) = 1 + px + qx^2$ with q and p constants, $q \neq 0$. Consider the discriminant of $s(x)$ equal zero, or, $\Delta = p^2 - 4q = 0$. Thus $q = \dfrac{p^2}{4}$ and the root $\lambda = -\dfrac{p}{2q}$ of $s(x)$ has multiplicity 2. Then, we obtain $1 + px + qx^2 = q(x-\lambda)^2$. This implies,

$$\begin{aligned} \mathbb{H}f(x) &= \frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{1 + px + qx^2} \\ &= \frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{q} \cdot \frac{1}{(x-\lambda)^2} \\ &= \frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{q\lambda^2} \cdot \frac{1}{\left(1 - \dfrac{x}{\lambda}\right)^2}. \end{aligned} \tag{14}$$

Let $\dfrac{1}{1 - \left(\dfrac{x}{\lambda}\right)} = \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{\lambda}\right)^n x^n$ be the geometric series. The convergence radius is given by $\left|\dfrac{x}{\lambda}\right| < 1$, or $|x| < |\lambda|$. Thus the function $g(x) = \dfrac{1}{1 - \left(\dfrac{x}{\lambda}\right)}$ is derivable in the interval $(-|\lambda|,|\lambda|)$. In addition, $g'(x) = \displaystyle\sum_{n=1}^{\infty} n \left(\frac{1}{\lambda}\right)^n x^{n-1}$ with convergence radius $\left|\dfrac{x}{\lambda}\right| < 1$. On the other side, since $g'(x) = \dfrac{1}{\lambda} \cdot \dfrac{1}{\left(1 - \dfrac{x}{\lambda}\right)^2}$, follows the result,

$$\frac{1}{\left(1 - \dfrac{x}{\lambda}\right)^2} = \sum_{n=0}^{\infty} (n+1) \cdot \frac{1}{\lambda^n} \cdot x^n. \tag{15}$$

By replacing Expression (15) in Expression (14) we get,

$$\begin{aligned} \mathbb{H}f(x) &= \frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{q\lambda^2} \cdot \sum_{n=0}^{\infty} (n+1) \cdot \frac{1}{\lambda^n} \cdot x^n \\ &= (\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x) \cdot \sum_{n=0}^{\infty} \frac{(n+1)x^n}{q\lambda^{n+2}}. \end{aligned} \tag{16}$$

The coefficient of x^n in (16) is given by,

$$\mathbb{H}V_n = \frac{\mathbb{H}V_0(n+1)}{q\lambda^{n+2}} + \frac{\mathbb{H}V_1 n}{q\lambda^{n+1}} + \frac{p\mathbb{H}V_0 n}{q\lambda^{n+1}}.$$

Based on the previous discussion, we obtain the following proposition.

Proposition 3.2. *Consider the homogeneous linear recurrence with constant coefficients* p, q, *where* $q \neq 0,$ *associated with the hybrid numbers* $(\mathbb{H}V_n)_{n\geq 0}$ *with initial conditions* $\mathbb{H}V_0$ *and* $\mathbb{H}V_1,$ (6)*, and equation* $\mathbb{H}V_{n+2} + p\mathbb{H}V_{n+1} + q\mathbb{H}V_n = 0, n \geq 0,$ (5)*. If the associated polynomial* $s(x) = 1 + px + qx^2$ *has discriminant* $\Delta = p^2 - 4q = 0,$ *then the explicit formula for* $\mathbb{H}V_n$ *is given by,*

$$\mathbb{H}V_n = \frac{\mathbb{H}V_0(n+1)}{q\lambda^{n+2}} + \frac{\mathbb{H}V_1 n}{q\lambda^{n+1}} + \frac{p\mathbb{H}V_0 n}{q\lambda^{n+1}}. \tag{17}$$

λ *is a root of* $s(x)$ *with multiplicity 2.*

Example 3.2. *Consider the numerical sequence* $(\mathbb{H}O_n)_{n\in\mathbb{N}}$ *given by,*

$$\begin{aligned}
&\mathbb{H}O_{n+2} - \mathbb{H}O_{n+1} + \frac{1}{4}\mathbb{H}O_n = 0, \;\; n \geq 0,\\
&\mathbb{H}O_0 = \frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h, \qquad (18)\\
&\mathbb{H}O_1 = \frac{1}{2} + \frac{1}{2}i + \frac{3}{8}\epsilon + \frac{1}{4}h.
\end{aligned}$$

By direct application of Proposition 3.2 with $s(x) = 1 - x + \frac{1}{4}x^2$ *and* $\lambda = 2$*, we obtain,*

$$\begin{aligned}
\mathbb{H}O_n &= \frac{\mathbb{H}O_0 \cdot (n+1)}{\frac{1}{4} \cdot 2^{n+2}} + \frac{\mathbb{H}O_1 \cdot n}{\frac{1}{4} \cdot 2^{n+1}} - \frac{\mathbb{H}O_0 \cdot n}{\frac{1}{4} \cdot 2^{n+1}}\\
\mathbb{H}O_n &= \frac{\left(\frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h\right)\cdot(n+1)}{2^n} + \frac{\left(\frac{1}{2} + \frac{1}{2}i + \frac{3}{8}\epsilon + \frac{1}{4}h\right)\cdot n}{2^{n-1}} + \frac{\left(\frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h\right)\cdot n}{2^{n-1}}\\
\mathbb{H}O_n &= \left(\frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h\right)\cdot\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h\right)\cdot n \cdot \left(\frac{1}{2}\right)^n\\
&+ \left(\frac{1}{2} + \frac{1}{2}i + \frac{3}{8}\epsilon + \frac{1}{4}h\right)\cdot n \cdot \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h\right)\cdot n \cdot \left(\frac{1}{2}\right)^{n-1}\\
\mathbb{H}O_n &= \left(\frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h\right)\cdot\left(\frac{1}{2}\right)^n + \left(1 + \frac{1}{2}i + \frac{1}{4}\epsilon + \frac{1}{8}h\right)\cdot n \cdot \left(\frac{1}{2}\right)^n.
\end{aligned}$$

3.3 The case $\mathbf{\Delta < 0}$ of polynomial $\mathbf{1 + px + qx^2}$

The approach for $\Delta < 0$ is not similar than the previous cases. Recall the Formula (7), namely,

$$\mathbb{H}f(x) = \frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{1 + px + qx^2} = \frac{\mathbb{H}V_0 + \mathbb{H}V_1 x + p\mathbb{H}V_0 x}{q\left(x^2 + \frac{p}{q}x + \frac{1}{q}\right)}.$$

The denominator $x^2+\frac{p}{q}x+\frac{1}{q}$, of generating function $\mathbb{H}f(x)$, has discriminant given by

$$\Delta' = \frac{p^2}{q^2} - \frac{4}{q} = \frac{p^2-4q}{q^2}.$$

Since $\Delta = p^2-4q < 0$ and $q^2 > 0$, then $\Delta' < 0$, and the roots of polynomial $x^2+\frac{p}{q}x+\frac{1}{q}$ are the following complex numbers,

$$\lambda = \frac{-\frac{p}{q} \pm \sqrt{\frac{p^2-4q}{q^2}}}{2} = \frac{-\frac{p}{q} \pm \frac{\sqrt{p^2-4q}}{|q|}}{2} = -\frac{p}{2q} \pm i\frac{\sqrt{4q-p^2}}{2q}.$$

Consider the notation

$$x^2 + \frac{p}{q}x + \frac{1}{q} = \left(x + \frac{p}{2q}\right)^2 - \frac{p^2}{4q^2} + \frac{1}{q} = \left(x + \frac{p}{2q}\right)^2 + \frac{1}{4}\left[\frac{-p^2+4q}{q^2}\right] = u^2 + A^2,$$

where $u = x + \frac{p}{2q}$ and $A^2 = \frac{1}{4}\left[\frac{-p^2+4q}{q^2}\right] > 0$. A long straightforward computation allows us to verify the following identity,

$$\frac{1}{x^2 + \frac{p}{q}x + \frac{1}{q}} = \sum_{j=0}^{\infty}\sum_{k=0}^{2j} \frac{(-1)^j}{A^{2j+2}} \binom{2j}{k} \left(\frac{p}{2q}\right)^{2j-k} x^k. \tag{19}$$

Then, the generating function for $(\mathbb{H}V_n)_{n\in\mathbb{N}}$ can be write under the form,

$$\begin{aligned} \mathbb{H}f(x) \quad &= \tfrac{\mathbb{H}V_0}{q} \sum_{j=0}^{\infty}\sum_{k=0}^{2j} \frac{(-1)^j}{A^{2j+2}} \tbinom{2j}{k} \left(\frac{p}{2q}\right)^{2j-k} x^k \\ &+ \left(\tfrac{\mathbb{H}V_1 + p\mathbb{H}V_0}{q}\right) \sum_{j=0}^{\infty}\sum_{k=0}^{2j} \frac{(-1)^j}{A^{2j+2}} \tbinom{2j}{k} \left(\frac{p}{2q}\right)^{2j-k} x^{k+1}, \end{aligned} \tag{20}$$

where $A^2 = \frac{1}{4}\left[\frac{-p^2+4q}{q^2}\right]$.

In addition, if $\lambda = -\frac{p}{2q} \pm i\frac{\sqrt{4q-p^2}}{2q} = \lambda_R \pm i\lambda_I$ are the roots of $x^2 + \frac{p}{q}x + \frac{1}{q}$, then

$u = x - \lambda_R$ and $A^2 = (\lambda_I)^2$. Thus, in terms of complex roots, we get

$$\mathbb{H}f(x) = \mathbb{H}V_0||\lambda||^2\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\frac{(-1)^{k+1}}{(\lambda_I^2)^{j+1}}\binom{2j}{k}(\lambda_R)^{2j-k}x^k$$
$$+ \left(\mathbb{H}V_1 - 2\lambda_R\mathbb{H}V_0\right)||\lambda||^2\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\frac{(-1)^{k+1}}{(\lambda_I^2)^{j+1}}\binom{2j}{k}(\lambda_R)^{2j-k}x^{k+1}.$$

The coefficient of x^n in $\mathbb{H}f(x)$ given by Expression (20) is the explicit formula for $\mathbb{H}V_n$, namely,

$$\mathbb{H}V_n = \frac{1}{q}\sum_{j=0}^{\infty}\frac{(-1)^j}{A^{2j+2}}\left(\frac{p}{2q}\right)^{2j-n}\left(\mathbb{H}V_0\binom{2j}{n} + (\mathbb{H}V_1 + p\mathbb{H}V_0)\binom{2j}{n-1}\left(\frac{p}{2q}\right)\right),$$

where $A^2 = \dfrac{1}{4}\left[\dfrac{-p^2+4q}{q^2}\right]$.

Similarly, the explicit formula of $\mathbb{H}V_n$ derived from generating function (20) is given by,

$$\mathbb{H}V_n = \mathbb{H}V_0||\lambda||^2\sum_{j=0}^{\infty}\frac{(-1)^{n+1}}{(\lambda_I^2)^{j+1}}\binom{2j}{n}(\lambda_R)^{2j-n}$$
$$+ \left(\mathbb{H}V_1 - 2\lambda_R\mathbb{H}V_0\right)||\lambda||^2\sum_{j=0}^{\infty}\frac{(-1)^{n}}{(\lambda_I^2)^{j+1}}\binom{2j}{n-1}(\lambda_R)^{2j-n+1}. \quad (21)$$

The previous discussion can be resumed in the following proposition,

Proposition 3.3. *Consider the homogeneous linear recurrence with constant coefficients p, q, where $q \neq 0$, associated with the hybrid numbers $(\mathbb{H}V_n)_{n\geq 0}$ with initial conditions $\mathbb{H}V_0$ and $\mathbb{H}V_1$,* (6)*, and equation $\mathbb{H}V_{n+2} + p\mathbb{H}V_{n+1} + q\mathbb{H}V_n = 0, n \geq 0,$* (5)*. If the associated polynomial $s(x) = x^2 + \frac{p}{q}x + \frac{1}{q}$ is such that $\Delta = \dfrac{p^2-4q}{q^2} < 0$, then the explicit formula for $\mathbb{H}V_n$ is given by,*

$$\mathbb{H}V_n = \mathbb{H}V_0||\lambda||^2\sum_{j=0}^{\infty}\frac{(-1)^{n+1}}{(\lambda_I^2)^{j+1}}\binom{2j}{n}(\lambda_R)^{2j-n}$$
$$+ \left(\mathbb{H}V_1 - 2\lambda_R\mathbb{H}V_0\right)||\lambda||^2\sum_{j=0}^{\infty}\frac{(-1)^{n}}{(\lambda_I^2)^{j+1}}\binom{2j}{n-1}(\lambda_R)^{2j-n+1},$$

where $\lambda = -\dfrac{p}{2q} \pm i\dfrac{\sqrt{4q-p^2}}{2q} = \lambda_R \pm i\lambda_I$ is root of $x^2 + \dfrac{p}{q}x + \dfrac{1}{q}$.

Example 3.3. *Consider the sequence given by $V_n = V_{n-2} - V_{n-1}, n \geq 2,$ and initial conditions $V_0 = 1, V_1 = 1$. The hybrid sequence associated is given by,*

$$\mathbb{H}V_{n+2} - \mathbb{H}V_{n+1} + \mathbb{H}V_n = 0 \tag{22}$$

for $n \geq 0,$ with initial conditions $\mathbb{H}V_0 = 1 + 2i + \varepsilon - h, \mathbb{H}V_1 = 2 + i - \varepsilon - 2h.$

The polynomial associated $s(x) = 1 - x + x^2$ has discriminate $\Delta < 0,$ then, by direct application of Proposition 3.3, the explicit formula for $\mathbb{H}V_n$ is given by,

$$\begin{aligned}\mathbb{H}V_n = \sum_{j=0}^{\infty} \frac{(-1)^j}{\left(\frac{3}{4}\right)^{j+1}} \binom{-1}{2}^{2j-n} &\left((1 + 2i + \varepsilon - h)\binom{2j}{n}\right. \\ &\left. + (2 + i - \varepsilon - 2h - 1 - 2i - \varepsilon + h)\binom{2j}{n}\binom{-1}{2}\right),\end{aligned}$$

$$\begin{aligned}\mathbb{H}V_n = \sum_{j=0}^{\infty} \frac{(-1)^j}{3^{j+1}} (-1)^{2j-n} 2^{2j+2} \frac{1}{2^{2j-n}} &\left((1 + 2i + \varepsilon - h)\binom{2j}{n}\right. \\ &\left. - \frac{1}{2}(1 - i - 2\varepsilon - h)\binom{2j}{n}\right),\end{aligned}$$

$$\mathbb{H}V_n = \sum_{j=0}^{\infty} \frac{(-1)^{3j-n}}{3^{j+1}} 2^{n+2} \left(\frac{1}{2} + \frac{5}{2}i + 2\varepsilon - \frac{1}{2}h\right)\binom{2j}{n}.$$

4 THE ANALYTIC FORMULA

The explicit formulas for the solutions of linear recurrence relations with constant coefficients are widely known in the literature. The analytic form is given by a linear combination of the powers of the roots of the characteristic polynomial, so-called Binet's formula. Consider the linear recurrence relation $a_{n+2} + pa_{n+1} + qa_n = 0, n \geq 0,$ where $q \neq 0,$ with initial conditions a_0 and a_1. The characteristic polynomial associated is $r(x) = x^2 + px + q$. In (Craveiro et al., 2022) was established the relationship of the polynomial $r(x)$ and the polynomial $s(x) = 1 + px + qx^2 = q\left(x^2 + \frac{p}{q}x + \frac{1}{q}\right)$ in the denominator of generating function (7), namely,

$$r(x) = x^2 + px + q = x^2\left[1 + \frac{1}{x}p + \frac{1}{x^2}q\right] = x^2 s\left(\frac{1}{x}\right), \tag{23}$$

for $\Delta > 0$ and $\Delta = 0$, and

$$r(x) = x^2 + px + q = x^2 s\left(\frac{1}{x}\right) = x^2 q\left[\frac{1}{q} + \frac{p}{qx} + \frac{1}{x^2}\right], \tag{24}$$

for $\Delta < 0$.

In addition, since $q \neq 0$, then if α is a root of $r(x)$ then $\alpha \neq 0$. Similarly, since $s(0) = 1 \neq 0$, then if λ is a root of $s(x)$ then $\lambda \neq 0$. Therefore, α is a root of $r(x)$ implies that $\frac{1}{\alpha}$ is a root of $s(x)$. Otherwise, if α is root of $s(x)$ then $\frac{1}{\alpha}$ is root of $r(x)$. By these observations and the direct application of Proposition 2.1 and Equations (23) -(24), we can derive the Binet's formula as well-know in the literature. In analogous way we can derive the result given in previous sections. In the next subsections each case are discussed and illustrative examples are given.

4.1 Binet's Formula in the cases $\Delta > 0$

Recall the recurrence relation (5) with initial conditions (6), namely, $\mathbb{H}V_n + p\mathbb{H}V_{n-1} + q\mathbb{H}V_{n-2} = 0$, and

$$\mathbb{H}V_0 = a_0 + a_1 i - (pa_1 + qa_0)\epsilon + [(p^2 - q)a_1 + pqa_0]h,$$

$$\mathbb{H}V_1 = a_1 - (pa_1 + qa_0)i + [(p^2 - q)a_1 + pqa_0)]\epsilon + [(-p^3 + 2pq)a_1 + (q^2 - p^2 q)a_0]h.$$

The characteristic polynomial associated with (5) is given by $r(x) = x^2 + px + q$. Suppose that the roots $x_1 \neq x_2$ of characteristic polynomial are simple. Thus $x_1 = \frac{-p+\sqrt{\Delta}}{2}$ and $x_2 = \frac{-p-\sqrt{\Delta}}{2}$, where $\Delta = p^2 - 4q$. The explicit formula for $\mathbb{H}V_n$ is given by,

$$\mathbb{H}V_n = C_1 x_1^n + C_2 x_2^n. \tag{25}$$

The constants C_1 and C_2 are determined by solving the Vandermonde system

$$\begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \mathbb{H}V_0 \\ \mathbb{H}V_1 \end{pmatrix}.$$

Then, we get $C_2 = \dfrac{\mathbb{H}V_1 - x_1\mathbb{H}V_0}{x_2 - x_1}$ and $C_1 = \dfrac{x_2\mathbb{H}V_0 - \mathbb{H}V_1}{x_2 - x_1}$. The preceding discussion is resumed in the following result,

Proposition 4.1. *(Binet's Formula for the case $\Delta > 0$) Consider the homogeneous linear recurrence with constant coefficients p, q, where $q \neq 0$, associated with the hybrid numbers $(\mathbb{H}V_n)_{n\geq 0}$ with initial conditions $\mathbb{H}V_0$ and $\mathbb{H}V_1$,* (6)*, and equation $\mathbb{H}V_{n+2} + p\mathbb{H}V_{n+1} + q\mathbb{H}V_n = 0, n \geq 0,$* (5)*, where is verified $\Delta = p^2 - 4q > 0$. Then the explicit formula of $\mathbb{H}V_n$ is given by*

$$\mathbb{H}V_n = \frac{x_2\mathbb{H}V_0 - \mathbb{H}V_1}{x_2 - x_1}x_1^n + \frac{\mathbb{H}V_1 - x_1\mathbb{H}V_0}{x_2 - x_1}x_2^n, \tag{26}$$

where $x_1 = \frac{-p+\sqrt{p^2-4q}}{2}$ and $x_2 = \frac{-p-\sqrt{p^2-4q}}{2}$.

By Expression (23), if $\lambda_1 \neq \lambda_2 \neq 0$ are roots of $s(x) = 1 + px + qx^2$ then $\frac{1}{\lambda_1} = x_1$ and $\frac{1}{\lambda_2} = x_2$ are roots of $r(x) = x^2 + px + q$. Since $\lambda_1\lambda_2 = \dfrac{1}{q}$ and $\lambda_1 + \lambda_2 = -\dfrac{p}{q}$, we obtain the equalities $\dfrac{\lambda_1}{\lambda_2} = -p\lambda_1 - 1$ and $\dfrac{\lambda_2}{\lambda_1} = -p\lambda_2 - 1$. Thus, multiplying the Equation (26) to $\dfrac{\lambda_1\lambda_2}{\lambda_1\lambda_2}$ and replacing $\frac{1}{\lambda_1} = x_1$ and $\frac{1}{\lambda_2} = x_2$ permit us to obtain the Expression (12), namely,

$$\mathbb{H}V_n = \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2}\left[\frac{\frac{1}{\lambda_2}\mathbb{H}V_0 - \mathbb{H}V_1}{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}\left(\frac{1}{\lambda_1}\right)^n + \frac{\mathbb{H}V_1 - \frac{1}{\lambda_1}\mathbb{H}V_0}{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}\left(\frac{1}{\lambda_2}\right)^n\right],$$

$$\mathbb{H}V_n = \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2}\left[\frac{\frac{1}{\lambda_2}\mathbb{H}V_0 - \mathbb{H}V_1}{\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}}\left(\frac{1}{\lambda_1}\right)^n + \frac{\mathbb{H}V_1 - \frac{1}{\lambda_1}\mathbb{H}V_0}{\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}}\left(\frac{1}{\lambda_2}\right)^n\right],$$

$$\mathbb{H}V_n = \lambda_1\lambda_2\left[\frac{\lambda_1\lambda_2(\frac{1}{\lambda_2}\mathbb{H}V_0 - \mathbb{H}V_1)}{\lambda_2(\lambda_1 - \lambda_2)}\left(\frac{1}{\lambda_1}\right)^{n+1} + \frac{\lambda_1\lambda_2(\mathbb{H}V_1 - \frac{1}{\lambda_1}\mathbb{H}V_0)}{\lambda_1(\lambda_1 - \lambda_2)}\left(\frac{1}{\lambda_2}\right)^{n+1}\right],$$

$$\mathbb{H}V_n = \lambda_1\lambda_2\left[\frac{\lambda_1\mathbb{H}V_0 - \lambda_1\lambda_2\mathbb{H}V_1}{\lambda_2(\lambda_1 - \lambda_2)}\left(\frac{1}{\lambda_1}\right)^{n+1} + \frac{\lambda_1\lambda_2\mathbb{H}V_1 - \lambda_2\mathbb{H}V_0}{\lambda_1(\lambda_1 - \lambda_2)}\left(\frac{1}{\lambda_2}\right)^{n+1}\right],$$

$$\mathbb{H}V_n = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\left[\left\{\frac{\lambda_1}{\lambda_2}\mathbb{H}V_0 - \lambda_1\mathbb{H}V_1\right\}\left(\frac{1}{\lambda_1}\right)^{n+1} + \left\{\lambda_2\mathbb{H}V_1 - \frac{\lambda_2}{\lambda_1}\mathbb{H}V_0\right\}\left(\frac{1}{\lambda_2}\right)^{n+1}\right],$$

$$\mathbb{H}V_n = \frac{-\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left[\{(-p\lambda_1 - 1)\mathbb{H}V_0 - \lambda_1\mathbb{H}V_1\}\left(\frac{1}{\lambda_1}\right)^{n+1} + \{\lambda_2\mathbb{H}V_1 - (-p\lambda_2 - 1)\mathbb{H}V_0\}\left(\frac{1}{\lambda_2}\right)^{n+1}\right],$$

$$\mathbb{H}V_n = \frac{-1}{q}\left[\frac{\mathbb{H}V_0 + \lambda_1\mathbb{H}V_1 + p\lambda_1\mathbb{H}V_0}{(\lambda_1 - \lambda_2)\lambda_1^{n+1}} + \frac{\mathbb{H}V_0 + \lambda_2\mathbb{H}V_1 + p\lambda_2\mathbb{H}V_0}{(\lambda_2 - \lambda_1)\lambda_2^{n+1}}\right].$$

Based on this discussion,

Proposition 4.2. *Consider the homogeneous linear recurrence with constant coefficients p, q, where $q \neq 0$, associated with the hybrid numbers $(\mathbb{H}V_n)_{n\geq 0}$ with initial conditions $\mathbb{H}V_0$ and $\mathbb{H}V_1$,* (6)*, and equation $\mathbb{H}V_{n+2} + p\mathbb{H}V_{n+1} + q\mathbb{H}V_n = 0, n \geq 0,$* (5)*, where $\Delta = p^2 - 4q > 0$. Then is verified the equality,*

$$\frac{\mathbb{H}V_0 x_2 - \mathbb{H}V_1}{x_2 - x_1} x_1^n + \frac{\mathbb{H}V_1 - x_1\mathbb{H}V_0}{x_2 - x_1} x_2^n = \frac{-1}{q}\left[\frac{\mathbb{H}V_0 + \lambda_1\mathbb{H}V_1 + p\lambda_1\mathbb{H}V_0}{(\lambda_1 - \lambda_2)\lambda_1^{n+1}} + \frac{\mathbb{H}V_0 + \lambda_2\mathbb{H}V_1 + p\lambda_2\mathbb{H}V_0}{(\lambda_2 - \lambda_1)\lambda_2^{n+1}}\right], \tag{27}$$

where $x_1 = \dfrac{1}{\lambda_1} = \frac{-p+\sqrt{p^2-4q}}{2}$ and $x_2 = \frac{-p-\sqrt{p^2-4q}}{2} = \dfrac{1}{\lambda_2}$.

Example 4.1. *Recall the Example 3.1 which consist of the explicit formula for the hybrid sequence associated with Fibonacci numbers, $\mathbb{H}F_n = \mathbb{H}F_{n-1} + \mathbb{H}F_{n-2}, n > 2$ with initial conditions $\mathbb{H}F_0 = i + \epsilon + 2h, \mathbb{H}F_1 = 1 + i + 2\epsilon + 3h$, given by,*

$$\begin{aligned}\mathbb{H}F_n = &\left[\frac{(i+\epsilon+2h) + (1+i+2\epsilon+3h)\left(\frac{-1-\sqrt{5}}{2}\right) - (i+\epsilon+2h)\left(\frac{-1-\sqrt{5}}{2}\right)}{-\sqrt{5}\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}}\right] \\ &+ \left[\frac{(i+\epsilon+2h) + (1+i+2\epsilon+3h)\left(\frac{-1+\sqrt{5}}{2}\right) - (i+\epsilon+2h)\left(\frac{-1+\sqrt{5}}{2}\right)}{\sqrt{5}\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}\right].\end{aligned}$$

The characteristic polynomial is given by $x^2 - x - 1 = 0$ with simple roots $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$. Then, the Binet's formula is given by,

$$\begin{aligned}\mathbb{H}F_n &= \frac{2 + (1+\sqrt{5})i + (3+\sqrt{5})\epsilon + (4+2\sqrt{5})h}{2\sqrt{5}}\,\frac{(1+\sqrt{5})^n}{2^n} \\ &+ \frac{2 + (1-\sqrt{5})i + (3-\sqrt{5})\epsilon + (4-2\sqrt{5})h}{-2\sqrt{5}}\,\frac{(1-\sqrt{5})^n}{2^n}\end{aligned}$$

Combining the Examples 3.1, 4.1 and the Proposition 4.2 we obtain the corollary below,

Corollary 4.1. *Consider the hybrid sequence associated with Fibonacci numbers, $\mathbb{H}F_n = \mathbb{H}F_{n-1} + \mathbb{H}F_{n-2}, n > 2$ with initial conditions $\mathbb{H}F_0 = i + \epsilon + 2h, \mathbb{H}F_1 = 1 + i + 2\epsilon + 3h$. Then, is verified the equality,*

$$\left[\frac{(i+\epsilon+2h) + (1+i+2\epsilon+3h)\left(\frac{-1-\sqrt{5}}{2}\right) - (i+\epsilon+2h)\left(\frac{-1-\sqrt{5}}{2}\right)}{-\sqrt{5}\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1}}\right]$$

$$+\left[\frac{(i+\epsilon+2h)+(1+i+2\epsilon+3h)\left(\frac{-1+\sqrt{5}}{2}\right)-(i+\epsilon+2h)\left(\frac{-1+\sqrt{5}}{2}\right)}{\sqrt{5}\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}\right]$$

$$=\frac{2+(1+\sqrt{5})i+(3+\sqrt{5})\epsilon+(4+2\sqrt{5})h}{2\sqrt{5}}\ \frac{(1+\sqrt{5})^n}{2^n}$$
$$+\frac{2+(1-\sqrt{5})i+(3-\sqrt{5})\epsilon+(4-2\sqrt{5})h}{-2\sqrt{5}}\ \frac{(1-\sqrt{5})^n}{2^n}.$$

4.2 Binet's Formula in the cases $\Delta=0$

Now, suppose that the characteristic polynomial has one root x_1 with multiplicity 2. Thus $x_1=\frac{-p}{2}$. The explicit formula for $\mathbb{H}V_n$ is given by,

$$\mathbb{H}V_n=C_1x_1^n+C_2nx_1^n. \tag{28}$$

The constants C_1 and C_2 are determined by solving the Vandermonde system

$$\begin{pmatrix}1 & 0\\ x_1 & x_1\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}=\begin{pmatrix}\mathbb{H}V_0\\ \mathbb{H}V_1\end{pmatrix}.$$

Then, we get $C_1=\mathbb{H}V_0$ and $C_2=\dfrac{\mathbb{H}V_1-x_1\mathbb{H}V_0}{x_1}$. The preceding discussion is resumed in the following result,

Proposition 4.3. *(Binet's Formula for the case $\Delta=0$) Consider the homogeneous linear recurrence with constant coefficients p,q, where $q\neq 0$, associated with the hybrid numbers $(\mathbb{H}V_n)_{n\geq 0}$ with initial conditions $\mathbb{H}V_0$ and $\mathbb{H}V_1$,* (6)*, and equation $\mathbb{H}V_{n+2}+p\mathbb{H}V_{n+1}+q\mathbb{H}V_n=0, n\geq 0,$* (5)*, where is verified $\Delta=p^2-4q=0$. Then the explicit formula of $\mathbb{H}V_n$ is given by*

$$\mathbb{H}V_n=(1-n)\mathbb{H}V_0x_1^n+n\mathbb{H}V_1x_1^{n-1}, \tag{29}$$

where $x_1=-\dfrac{p}{2}$.

The Expression (23) allow us to observe that if $\lambda_1\neq 0$ is a root of $s(x)=1+px+qx^2$ then $\dfrac{1}{\lambda_1}=x_1$ is a root of $r(x)=x^2+px+q$. Since $\lambda_1^2=\dfrac{1}{q}$ and $2\lambda_1=-\dfrac{p}{q}$. Thus, multiplying the Equation (29) by $\dfrac{\lambda_1^2}{\lambda_1^2}$ and substituting $\dfrac{1}{\lambda_1}=x_1$ we obtain the Expression (17), i.e,

$$\mathbb{H}V_n=\frac{\lambda_1^2}{\lambda_1^2}\left[(1-n)\mathbb{H}V_0\left(\frac{1}{\lambda_1}\right)^n+n\mathbb{H}V_1\left(\frac{1}{\lambda_1}\right)^{n-1}\right],$$

$$\mathbb{H}V_n = \lambda_1^2 \left[(1-n)\mathbb{H}V_0 \left(\frac{1}{\lambda_1}\right)^{n+2} + n\mathbb{H}V_1 \left(\frac{1}{\lambda_1}\right)^{n+1} \right],$$

$$\mathbb{H}V_n = \lambda_1^2 \left[(1+n)\mathbb{H}V_0 \left(\frac{1}{\lambda_1}\right)^{n+2} - \frac{2\mathbb{H}V_0 n}{\lambda_1} \left(\frac{1}{\lambda_1}\right)^{n+1} + n\mathbb{H}V_1 \left(\frac{1}{\lambda_1}\right)^{n+1} \right],$$

$$\mathbb{H}V_n = \frac{\mathbb{H}V_0(n+1)}{q\lambda_1^{n+2}} + \frac{\mathbb{H}V_1 n}{q\lambda_1^{n+1}} + \frac{p\mathbb{H}V_0 n}{q\lambda_1^{n+1}}.$$

Under previous discussion, we have the identity,

Proposition 4.4. *Consider the homogeneous linear recurrence with constant coefficients p, q, where $q \neq 0$, associated with the hybrid numbers $(\mathbb{H}V_n)_{n\geq 0}$ with initial conditions $\mathbb{H}V_0$ and $\mathbb{H}V_1$,* (6)*, and equation $\mathbb{H}V_{n+2}+p\mathbb{H}V_{n+1}+q\mathbb{H}V_n = 0, n \geq 0,$* (5)*, where is verified $\Delta = p^2-4q = 0.$ Then is verified the equality,*

$$(1-n)\mathbb{H}V_0 x_1^n + n\mathbb{H}V_1 x_1^{n-1} = \frac{\mathbb{H}V_0(n+1)}{q\lambda_1^{n+2}} + \frac{\mathbb{H}V_1 n}{q\lambda_1^{n+1}} + \frac{p\mathbb{H}V_0 n}{q\lambda_1^{n+1}}, \tag{30}$$

where $x_1 = -\frac{p}{2} = \frac{1}{\lambda_1}$.

Below, we provide an illustrative example.

Example 4.2. *The Oresme sequence is defined by the homogeneous linear recurrence $O_n = O_{n-1} - \frac{1}{4}O_{n-2}$, for $n \geq 2$, and initial conditions $O_0 = 0$ and $O_1 = \frac{1}{2}$. The hybrid sequence associated with Oresme numbers is defined by,*

$$\mathbb{H}O_n = \mathbb{H}O_{n-1} - \frac{1}{4}\mathbb{H}O_{n-2},$$

for $n \geq 2$, and initial conditions $\mathbb{H}O_0 = \frac{1}{2}i + \frac{3}{8}\epsilon + \frac{1}{4}h, \mathbb{H}O_1 = \frac{1}{2} + \frac{1}{2}i + \frac{3}{8}\epsilon + \frac{1}{4}h.$

The characteristic polynomial associated with the hybrid sequence is given by $p(x) = x^2 - x + \frac{1}{4} = 0$. Since $p(x)$ has one root with multiplicity 2, $x_1 = \frac{1}{2}$, the Binet's formula for the hybrid numbers $\mathbb{H}O_n, n \in \mathbb{N}$, is given by

$$\mathbb{H}O_n = \left(\frac{1}{2}i + \frac{1}{2}\epsilon + \frac{3}{8}h\right)\left(\frac{1}{2}\right)^n + \left(1 + \frac{1}{2}i + \frac{1}{4}\epsilon + \frac{1}{8}h\right) n \left(\frac{1}{2}\right)^n \tag{31}$$

4.3 Binet's Formula in the cases $\Delta < 0$

The solution in this case depend on sine and cosine functions. However, these studies offer concise treatments, with few examples in the literature. Given the recurrence relation (5) with initial conditions (6), the characteristic polynomial associated with (5) is given by $r(x) = x^2 + px + q$. Suppose that the roots $x_1 \neq x_2$ of characteristic polynomial are complex numbers, or $\Delta = p^2 - 4q < 0, q > 0$. The roots are given under the trigonometric form, $x_1 = \rho[\cos(\theta) + i\sin(\theta)]$ and $x_2 = \rho[\cos(\theta) - i\sin(\theta)]$, where $\rho = \sqrt{\left(-\frac{p}{2}\right)^2 + \left(\frac{\sqrt{4q-p^2}}{2}\right)^2} = \sqrt{q}$ and θ is an angle such that $\cos(\theta) = -\frac{p}{2\sqrt{q}}$ and $\sin(\theta) = \frac{\sqrt{4q-p^2}}{2\sqrt{q}}$. Then the explicit formula for $\mathbb{H}V_n$ is given by,

$$\mathbb{H}V_n = \rho^n[C_1\cos(n\theta) + C_2\sin(n\theta)]. \tag{32}$$

The constants C_1 and C_2 are determined by solving the Vandermonde system

$$\begin{pmatrix} 1 & 0 \\ \rho\cos(\theta) & \rho\sin(\theta) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \mathbb{H}V_0 \\ \mathbb{H}V_1 \end{pmatrix}.$$

The preceding discussion is resumed in the following result,

Proposition 4.5. *(Binet's Formula for the case $\Delta < 0$) Consider the homogeneous linear recurrence with constant coefficients p, q, where $q > 0$, associated with the hybrid numbers $(\mathbb{H}V_n)_{n\geq 0}$ with initial conditions $\mathbb{H}V_0$ and $\mathbb{H}V_1$,* (6)*, and equation $\mathbb{H}V_{n+2} + p\mathbb{H}V_{n+1} + q\mathbb{H}V_n = 0, n \geq 0,$* (5)*, where is verified $\Delta = p^2 - 4q < 0$. Then the explicit formula of $\mathbb{H}V_n$ is given by*

$$\mathbb{H}V_n = \rho^n[C_1\cos(n\theta) + C_2\sin(n\theta)]. \tag{33}$$

where $\rho = \sqrt{q}$ and θ is an angle such that $\cos(\theta) = -\frac{p}{2\sqrt{q}}$ and $\sin(\theta) = \frac{\sqrt{4q-p^2}}{2\sqrt{q}}$, and the constants C_1 and C_2 are determined by solving the Vandermonde system.

Example 4.3. *To clarify, let the homogeneous linear recurrence with constant coefficients, $\mathbb{H}V_n - \mathbb{H}V_{n-1} + \mathbb{H}V_{n-2} = 0$, for $n \geq 2$, and initial conditions $\mathbb{H}V_0, \mathbb{H}V_1$. The characteristic polynomial, given by $x^2 - x + 1 = 0$, has the roots are $x_1 = \frac{1+i\sqrt{3}}{2}$ and $x_2 = \frac{1-i\sqrt{3}}{2}$. In* *addition,* $\rho = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ *and* $\theta = \frac{\pi}{3}$.

Then, we obtain

$$\mathbb{H}V_n = C_1 \cos\left(\frac{n\pi}{3}\right) + C_2 \sin\left(\frac{n\pi}{3}\right),$$

where $C_1 = \mathbb{H}V_0$ *and* $C_2 = \frac{2\mathbb{H}V_1 - \mathbb{H}V_0}{\sqrt{3}}$.

Recall that the roots of the polynomial $\frac{1}{q}s(x) = t(x) = x^2 + \frac{p}{q}x + \frac{1}{q}$ are the following complex numbers,

$$\lambda = \frac{-\frac{p}{q} \pm \sqrt{\frac{p^2-4q}{q^2}}}{2} = \frac{-\frac{p}{q} \pm \frac{\sqrt{p^2-4q}}{|q|}}{2} = -\frac{p}{2q} \pm i\frac{\sqrt{4q-p^2}}{2q},$$

and since $q \neq 0$, λ is a root of $s(x)$ if and only if λ is a root of $t(x)$.

By Expression (24), if $\lambda_1 \neq \lambda_2 \neq 0$ are roots of $s(x) = 1 + px + qx^2$ then $\lambda_1 = \frac{x_1}{q}$ and $\lambda_2 = \frac{x_2}{q}$ where x_1 and x_2 are the roots of characteristic polynomial $r(x) = x^2 + px + q$. And $\lambda_1\lambda_2 = \frac{1}{q}$.

In addition, if $x = -\frac{p}{2} \pm i\frac{\sqrt{4q-p^2}}{2} = x_R \pm ix_I$ is a root of $x^2 + px + q$ then, $x_R = q\lambda_R$ and $x_I = q\lambda_I$, $u = x - \frac{1}{q}x_R$ and $A^2 = \left(\frac{x_I}{q}\right)^2$, where $A^2 = \frac{1}{4}\left[\frac{-p^2+4q}{q^2}\right]$.

Thus, by the Expression (20), in terms of complex roots of characteristic polynomial $r(x) = x^2 + px + q$, we get,

$$\mathbb{H}f(x) = \frac{\mathbb{H}V_0}{||x||^2}\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\frac{(-1)^{k+1}}{\left(\left(\frac{x_I}{q}\right)^2\right)^{j+1}}\binom{2j}{k}\left(\frac{x_R}{q}\right)^{2j-k}x^k \tag{34}$$

$$+\frac{(\mathbb{H}V_1 - 2x_R\mathbb{H}V_0)}{||x||^2}\sum_{j=0}^{\infty}\sum_{k=0}^{2j}\frac{(-1)^{k+1}}{\left(\left(\frac{x_I}{q}\right)^2\right)^{j+1}}\binom{2j}{k}\left(\frac{x_R}{q}\right)^{2j-k}x^{k+1}.$$

Then, the coefficient of x^n in $\mathbb{H}f(x)$ in (34) is the explicit formula for $\mathbb{H}V_n$, namely,

$$\mathbb{H}V_n = \frac{\mathbb{H}V_0}{||x||^2}\sum_{j=0}^{\infty}\frac{(-1)^{n+1}}{\left(\left(\frac{x_I}{q}\right)^2\right)^{j+1}}\binom{2j}{n}\left(\frac{x_R}{q}\right)^{2j-n} \tag{35}$$

$$+\frac{(\mathbb{H}V_1-2x_R\mathbb{H}V_0)}{||x||^2}\sum_{j=0}^{\infty}\frac{(-1)^n}{\left(\left(\frac{x_I}{q}\right)^2\right)^{j+1}}\binom{2j}{n-1}\left(\frac{x_R}{q}\right)^{2j-(n-1)}.$$

$$\mathbb{H}V_n=\frac{\mathbb{H}V_0}{\rho^2}\sum_{j=0}^{\infty}\frac{(-1)^{n+1}}{\left(\left(\frac{\rho\sin(\theta)}{q}\right)^2\right)^{j+1}}\binom{2j}{n}\left(\frac{\rho\cos(\theta)}{q}\right)^{2j-n}$$
$$+\frac{(\mathbb{H}V_1-2\rho\cos(\theta)\mathbb{H}V_0)}{\rho^2}\sum_{j=0}^{\infty}\frac{(-1)^n}{\left(\left(\frac{\rho\sin(\theta)}{q}\right)^2\right)^{j+1}}\binom{2j}{n-1}\left(\frac{\rho\cos(\theta)}{q}\right)^{2j-(n-1)}. \tag{36}$$

Proposition 4.6. *Consider the homogeneous linear recurrence with constant coefficients* p, q, *where* $q > 0$, *associated with the hybrid numbers* $(\mathbb{H}V_n)_{n\geq 0}$ *with initial conditions* $\mathbb{H}V_0$ *and* $\mathbb{H}V_1$, (6)*, and equation* $\mathbb{H}V_{n+2}+p\mathbb{H}V_{n+1}+q\mathbb{H}V_n=0, n\geq 0$, (5)*, where is verified* $\Delta=p^2-4q<0$. *Then*

$$\rho^n[C_1\cos(n\theta)+C_2\sin(n\theta)]=\frac{\mathbb{H}V_0}{\rho^2}\sum_{j=0}^{\infty}\frac{(-1)^{n+1}}{\left(\left(\frac{\rho\sin(\theta)}{q}\right)^2\right)^{j+1}}\binom{2j}{n}\left(\frac{\rho\cos(\theta)}{q}\right)^{2j-n}$$
$$+\frac{(\mathbb{H}V_1-2\rho\cos(\theta)\mathbb{H}V_0)}{\rho^2}\sum_{j=0}^{\infty}\frac{(-1)^n}{\left(\left(\frac{\rho\sin(\theta)}{q}\right)^2\right)^{j+1}}\binom{2j}{n-1}\left(\frac{\rho\cos(\theta)}{q}\right)^{2j-(n-1)}, \tag{37}$$

where $\rho=\sqrt{q}$ *and* θ *is an angle such that* $\cos(\theta)=\frac{p}{2\sqrt{q}}$ *and* $\sin(\theta)=\frac{\sqrt{4q-p^2}}{2\sqrt{q}}$, *and the constants* C_1 *and* C_2 *are determined by solving the Vandermonde system.*

Example 4.4. *Let the homogeneous linear recurrence with constant coefficients,* $\mathbb{H}V_n-\mathbb{H}V_{n-1}+\mathbb{H}V_{n-2}=0$, *for* $n\geq 2$, *and initial conditions* $\mathbb{H}V_0, \mathbb{H}V_1$. *Then, is verified,*

$$\sum_{j=0}^{\infty}\frac{(-1)^{3j-n}}{3^{j+1}}2^{n+2}\left(\frac{1}{2}+\frac{5}{2}i+2\varepsilon-\frac{1}{2}h\right)\binom{2j}{n}=C_1\cos\left(\frac{n\pi}{3}\right)+C_2\sin\left(\frac{n\pi}{3}\right)$$

where $C_1=\mathbb{H}V_0$ *and* $C_2=\frac{2\mathbb{H}V_1-\mathbb{H}V_0}{\sqrt{3}}$.

5 CONCLUSION

This work presented a suggestion of using a counting tool, the ordinary generating function; namely, a power series for obtaining formulas explicitly for solving a hybrid homogeneous linear recurrence with constants of order 2 with initial values, without the need to solve a Vandermonde system. With the mastery of the techniques guaranteed in this work, it is enough for the student to know how to interpret and model the problem, since the solution will appear naturally in the coefficients of the polynomials that model them, which, in particular, is a series of powers.

In general, given the fact that most textbooks contain information on Algebra and Geometry, using algebraic knowledge about functions and polynomials to solve counting problems can also be a teaching approach. Specifically, with the use of numerical sequences, content that must be perfectly understood in the classroom according to the National Common Curricular Base - BNCC, the approach of this article represents another approach to understand and deepen the process of creation of combinatorial thinking, which is essential for understanding other mathematical concepts, such as probabilities as a function of the cardinality quotient of events, a subject also described in the National Curricular Common Base-BNCC.

Finally, the content of the previous sections shows that the mathematical tools, put into play around the generating function, also constitute a mathematical culture for the teacher. This will allow him to better visualize the close links between combinatorics and other areas of mathematics.

ACKNOWLEDGEMENTS

The authors express their sincere thanks to the Universidade Federal da Grande Dourados-UFGD/MEC-Brazil and to the Universidade Federal de Mato Grosso do Sul – UFMS/MEC – Brazil for their valuable support.

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How to cite this article

Wagner, A., Ribeiro, A. C., Spreafico, E. V. P., Craveiro, I. M., & Carlos, T. B. (2025). On second-order linear difference equations associated with hybrid sequences and generating functions. *Ciência e Natura*, Santa Maria, v. 47, spe. 1, e84462. DOI: https://doi.org/10.5902/2179460X84462.