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Mathematics

# Codimension one bifurcations of discontinuous vector fields - the boundary saddle and node cases

Bifurcações de campos vetoriais descontínuos de codimensão um - os casos sela e nó no bordo

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## ABSTRACT

We use the method of regularization of discontinuous vector fields, as described by (Sotomayor & Teixeira, 1998), to explain, in terms of the classical smooth bifurcation, two codimension one bifurcations of families of discontinuous vector fields generated by the collision of a saddle with the discontinuity set, and the collision of a node with the discontinuity set. These bifurcations are contained in the list presented both by in (Filippov, 1998), and by in (Kuznetsov et al., 2003).

**Keywords:** Bifurcation; Regularization; Discontinuous vector field

## RESUMO

Utilizamos o método de regularização de campos vetoriais descontínuos, descrito por (Sotomayor & Teixeira, 1998), para explicar, em termos das bifurcações clássicas, duas bifurcações de codimensão, um de famílias de campos vetoriais descontínuos gerados pela colisão de uma sela com o conjunto de descontinuidade, e outra a colisão de um nó com o mesmo conjunto. Essas bifurcações estão contidas em uma lista apresentada igualmente por (Filippov, 1998), e por (Kuznetsov et al., 2003).

**Palavras-chave:** Bifurcações; Regularização; Campos vetoriais descontínuos

# 1 INTRODUCTION

The classical theory of bifurcations of smooth vector fields is well understood since the second half of last century. On the other hand, we still have some important open problems concerning bifurcations of discontinuous vector fields, (Colombo et al., 2012). So, it is relevant to know the bifurcation theory of discontinuous vector fields applied to the real cases, see (Andronov et al., 1966), (Di Bernardo et al., 2008), (Jeffrey, 2020) and (Jeffrey, 2018). There are various approaches to increase the knowledge of these systems, and one of these is the regularization method, which we choose to our work. A interesting problem is the explanation of bifurcations in discontinuous vector fields, through the well-known bifurcations in smooth vector fields, using the Sotomatyor-Teixeira regularization method, see (Sotomayor & Teixeira, 1998). Our aim is to work in this problem considering saddle and node bifurcations of discontinuous vector fields. For our approach, we use a specific regularization function that allow us to make some simple calculations. Recently, in (Buzzi & Santos, 2019), the authors proposed studying bifurcations that occur when discontinuous vector fields contain a saddle-fold singularity using the regularization method. For the sake of clarity let us introduce the notions and results needed to our work.

Let $M$ be a compact and connected subset of the plane $\mathbb{R}^2$ and $F : M \to \mathbb{R}$ be a $C^\infty$ function having $0$ as a regular value. We suppose that the plane is provided with the usual metric. We assume, for simplicity's sake, that $D = F^{-1}(0)$ has a single connected component so that $M \setminus D$ consists of exactly two connected components, denoted by $N = F^{-1}\big([0,\infty)\big)$ and $S = F^{-1}\big((-\infty,0]\big)$.

The space of $C^r$ vector fields defined on $M$, for $r \geq 1$, will be denoted by $\mathfrak{X}^r(M)$.

Let $\Omega^r(M) = \Omega^r(M,F)$, $r \geq 1$, be the space of the vector fields $Z$ defined on $M$, as follows:

$$Z(p) = \begin{cases} X(p), & F(p) \geq 0 \\ Y(p), & F(p) \leq 0 \end{cases}$$

where $X,\, Y \in \mathfrak{X}^r(M)$. If $Z \in \Omega^r(M)$, we write $Z = (X,Y)$. The elements of $\Omega^r(M)$ are called *discontinuous vector fields*. Notice that if $Z \in \Omega^r(M)$ then it is bivalued along $D$, where the set $D$ is called *discontinuity set.* For applications of the theory of discontinuous vector fields we recommend Andronov et al. (1966), Di Bernardo et al. (2008), Kuznetsov et al. (2003) and their references.

In the discontinuity set of $Z \in \Omega^r(M)$, we have three subsets depending on the intersection of the orbits of $X$ and $Y$ with $D$. Following Filippov terminology, we distinguish these sets by

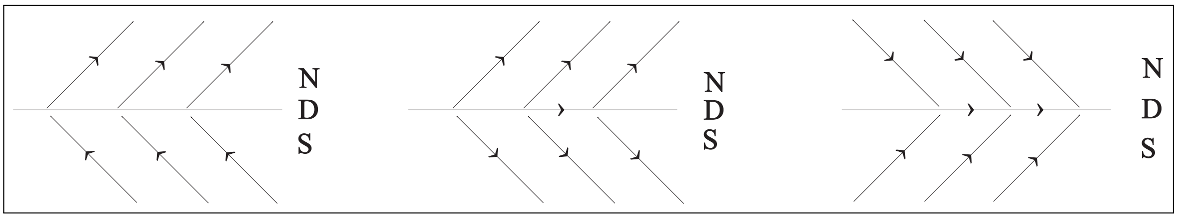
(SW) Sewing Arc: characterized by $(XF)(YF)(p) > 0$,

(ES) Escaping Arc: characterized by $XF(p) > 0$ and $YF(p) < 0$,

(SL) Sliding Arc: characterized by $XF(p) < 0$ and $YF(p) > 0$.

In figure 1 we have each one of these sets, respectively.

Figure 1 – Sewing, escape and sliding sets




Source: the authors (2023)

As usual, $XF$ will denote the derivative of function $F$ in the direction of the vector field $X : XF = dF \cdot X$, where the dot denotes the usual inner product on the plane.

On the arcs ES or SL, we define the *Filippov vector field,* $F_Z$, associated to $Z = (X, Y)$, as follows: if $p \in$ SL or $p \in$ ES, then $F_Z(p)$ denotes the vector tangent to $D$ contained in the cone spanned by $X(p)$ and $Y(p)$. From this definition, the expression of $F_Z$ associated to the discontinuous vector field $Z$ is

$$F_Z(x, y) = lX(x, y) + (1 - l)Y(x, y) \tag{1}$$

where

$$l = \frac{\nabla F(x, y) \cdot Y(x, y)}{\nabla F(x, y) \cdot \big(Y(x, y) - X(x, y)\big)}.$$

More details can be found in Filippov (1998) or Sotomayor & Machado (2002).

A point $p \in D$ is called a *critical point* of the Filippov vector field $F_Z$ if $XF(p)YF(p) < 0$ and det $[X,\, Y](p) = 0$. Here $[X,\, Y]$ is the matrix whose lines are the expressions of the coordinates of the vector fields $X$ and $Y$, in this order. If the derivative of $(\det[X, Y]|_D)$ at point $p$ is not null, we say that $p$ is a *hyperbolic critical point* of $F_Z$.

A hyperbolic critical point $p$ of the Filippov vector field $F_Z$ is a *saddle* if either $p$ is in the sliding set and is a repelling point (i.e., satisfies $d(\det[X, Y]|_D)(p) > 0$) or $p$ belongs to the escaping set and is an attracting point (i.e., $d(\det[X, Y]|_D)(p) < 0$). An *attracting node* (or *stable node*) $p$ is a hyperbolic critical point of $F_Z$ if $p$ that belongs to sliding set and $d(\det[X, Y]|_D)(p) < 0$. On the other hand, if $p$ belongs to the escaping set and $d(\det[X,Y]|_D)(p) > 0$ we call $p$ a *repelling node* of $F_Z$. The definition of other types of singularities of the Filippov vector field can be found in Sotomayor & Machado (2002).

Let $Z_\lambda = (X_\lambda, Y_\lambda) \in \Omega^r$, defined in $M$, a one-parameter family of discontinuous vector fields with $\lambda$ a parameter in a small interval around the origin. Our goal is to analyze the bifurcations that occur with the presence of a saddle or a node for the north set family of vector fields, $X_\lambda$, for small and negative values of the parameter, that collides with the origin of the plane when the parameter vanishes. Other types of bifurcation occur when we change the type of singularity of the family of vector fields in the north set, the interesting case of a focus is still in preparation to a forthcoming work by the authors. All this analysis is based on a slight modification of the regularization method of discontinuous vector fields presented by Sotomayor and Teixeira, see Sotomayor & Teixeira (1998). Our version of the regularization method is presented below.

Let $\varepsilon > 0$, we define the *regularization*, of $Z = (X, Y) \in \Omega^r(M)$ as the one parameter family of vector fields $Z_\varepsilon \in \mathfrak{X}^r(M)$ given by

$$Z_\varepsilon(p) = \big(1 - \varphi_\varepsilon(F(p))\big)Y(p) + \varphi_\varepsilon(F(p))X(p) \tag{2}$$

where $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$. The function $\varphi : \mathbb{R} \to \mathbb{R}$, of class $C^\infty$, is called *transition function* and, from now on, we consider

$$\varphi(x) = \frac{1}{2} + \frac{x}{2\sqrt{x^2+1}}. \tag{3}$$

The variable $\varepsilon$ is called the *regularization parameter*. If $Z_\lambda = (X_\lambda, Y_\lambda)$ and $\varepsilon > 0$, the regularization of this family of discontinuous vector fields is the two-parameter family of smooth vector fields given by

$$Z_{\lambda,\varepsilon}(p) = \big(1 - \varphi_\varepsilon(F(p))\big)Y_\lambda(p) + \varphi_\varepsilon(F(p))X_\lambda(p) \tag{4}$$

with $p \in M$.

We notice that the function (3) is not a transition function as defined in Sotomayor & Teixeira (1998), but for $\varepsilon > 0$ if we define the family of functions $\varphi_\varepsilon$ then this family has the Heaviside function as a limit, when $\varepsilon$ goes to zero. And this fact is the principal property of the transition functions as defined by Sotomayor and Teixeira.

The next propositions give us interesting facts about the regularization using the transition function (3) and the definitions and conventions above.

**Proposition 1.** *Let $p \in D$ be a hyperbolic singularity of the vector field $F_Z$, where $Z = (X, Y) \in \Omega^r(M)$. Then, there exists a neighborhood $V$ of $p$ in $M$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the respective regularized family of vector fields $Z_\varepsilon$, given by* (2)*, has only one singularity $p_\varepsilon \in V$ which is hyperbolic and is a saddle or node as $p$ is a saddle or node for $F_Z$.*

**Proposition 2.** *Let $Z = (X, Y) \in \Omega^r(M)$. Let $p \in D$ be a quadratic tangency of an orbit of $Z$ with the discontinuous set $D$. Then, there exists a neighborhood $V$ of $p$ in $M$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the respective regularized family of vector fields $Z_\varepsilon$, given by* (2)*, does not have any singularity on $V$.*

The proof of these propositions are a slight modifications of the respective result of Sotomayor & Machado (2002), and we will omit they.

Various types of codimension one bifurcations for discontinuous vector fields on the plane where presented initially in the book of Filippov (1998). In Kuznetsov et al. (2003) presented the diagram for each one of these codimension one bifurcations. Our aim here is first characterizes each of the mentioned cases, then we regularize and explain in terms of the regular bifurcations each one of these five types of regularization: two node boundary and three saddle boundary.

To obtain our results, this work is divided in the following manner. What happens in the regularization of a discontinuous vector field which has singularities or folds? This question will be answered in Section 2. These results will lead us to the understanding of the bifurcations above mentioned. In Section 2 we present the expression of a family of discontinuous vector fields $Z_\lambda = (X_\lambda, Y_\lambda)$ with a singularity for the vector field in the north set and assuming that vector field in the south set is constant. We also present the expression for the regularization of this family of discontinuous vector fields. In section 3 first we show the bifurcations of type saddle and node collision, following Filippov (1998) and Kuznetsov et al. (2003), secondly, we obtain relations on the coefficients of the family of discontinuous vector fields, $Z_\lambda$, that characterizes each type of saddle and node boundary bifurcations. The regularization and preliminary results for these families, for each type of discontinuous vector field, are presented in Section 4. In Sections 5 and 6 we present, respectively, the explanation, in terms of smooth bifurcations, for saddle and node bifurcations of discontinuous vector fields.

## 2 FAMILY OF DISCONTINUOUS VECTOR FIELDS

For our purposes we will obtain the expression for a family of discontinuous vector fields with a singularity in the north set which collides with the origin of the plane, when the parameter vanish, and supposing that the vector field on the south set is constant. To do this, let $Z_\lambda = (X_\lambda, Y_\lambda) \in \Omega^r(M)$ be a one-parameter family of $C^r, r \geq 2,$ discontinuous vector fields with $\lambda \in J,$ where $J$ is some small interval around the origin.

Let $X_{\tilde{\lambda}}(\tilde{x},\, \tilde{y}) = \big(X_1(\tilde{x},\, \tilde{y},\, \tilde{\lambda}),\, X_2(\tilde{x},\, \tilde{y},\, \tilde{\lambda})\big)$ be a one-parameter family of vector fields of class $C^2$ on $M$. By the Taylor series expansion in a neighborhood of the origin, the family $X_{\tilde{\lambda}}$, when the parameter vanishes, has the following expression

$$\tilde{x}' = \frac{\partial X_1}{\partial \tilde{x}}(0,\,0,\,0)\tilde{x} + \frac{\partial X_1}{\partial \tilde{y}}(0,\,0,\,0)\tilde{y} + \frac{\partial X_1}{\partial \tilde{\lambda}}(0,\,0,\,0)\tilde{\lambda} + \tilde{R}_2^1(\tilde{x},\,\tilde{y},\,\tilde{\lambda})$$
$$\tilde{y}' = \frac{\partial X_2}{\partial \tilde{x}}(0,\,0,\,0)\tilde{x} + \frac{\partial X_2}{\partial \tilde{y}}(0,\,0,\,0)\tilde{y} + \frac{\partial X_2}{\partial \tilde{\lambda}}(0,\,0,\,0)\tilde{\lambda} + \tilde{R}_2^2(\tilde{x},\,\tilde{y},\,\tilde{\lambda})$$

where $\tilde{R}_2^1(\tilde{x},\,\tilde{y},\,\tilde{\lambda})$ and $\tilde{R}_2^2(\tilde{x},\,\tilde{y},\,\tilde{\lambda})$ are remainders given by

$$\begin{aligned} \tilde{R}_2^1(\tilde{x},\,\tilde{y},\,\tilde{\lambda}) &= X^1_{2,0,0}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{x}^2 + X^1_{0,2,0}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{y}^2 + X^1_{0,0,2}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{\lambda}^2 \\ &\quad + X^1_{1,1,0}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{x}\tilde{y} + X^1_{1,0,1}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{x}\tilde{\lambda} + X^1_{0,1,1}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{y}\tilde{\lambda} \\ \tilde{R}_2^2(\tilde{x},\,\tilde{y},\,\tilde{\lambda}) &= X^2_{2,0,0}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{x}^2 + X^2_{0,2,0}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{y}^2 + X^2_{0,0,2}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{\lambda}^2 \\ &\quad + X^2_{1,1,0}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{x}\tilde{y} + X^2_{1,0,1}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{x}\tilde{\lambda} + X^2_{0,1,1}(\tilde{x},\,\tilde{y},\,\tilde{\lambda})\tilde{y}\tilde{\lambda} \end{aligned} \tag{5}$$

where $X^i_{j,k,l}$ for $i = 1,\,2,$ and $j, k, l = 0,\,1,\,2,$ are functions defined in a neighborhood of the origin given by the Theorem of Taylor.

For simplicity's sake, we write the expression of family $X_{\tilde{\lambda}}$ as

$$\tilde{x}' = a\tilde{x} + b\tilde{y} + e\tilde{\lambda} + \tilde{R}_2^1$$
$$\tilde{y}' = c\tilde{x} + d\tilde{y} + f\tilde{\lambda} + \tilde{R}_2^2$$

where $a,\, b,\, c,\, d,\, e,\, f$ are real numbers and $\tilde{R}_2^1,\, \tilde{R}_2^2$ are remainders given by (5).

Family $Y_{\tilde{\lambda}}$ must be a constant vector field pointing to the north set. By the Tubular Flow Theorem, let us suppose that $Y_{\tilde{\lambda}} = (0,\, \tilde{\mu})$ where $\tilde{\mu}$ is a real number.

**Proposition 3.** *Let $\tilde{Z}_{\tilde{\lambda}} = \big(\tilde{X}_{\tilde{\lambda}}, \tilde{Y}_{\tilde{\lambda}}\big)$ be a one-parameter family of discontinuous vector fields defined in $M$ where $\tilde{X}_{\tilde{\lambda}}$ and $\tilde{Y}_{\tilde{\lambda}}$ are $C^r$ vector fields, $r \geq 2$, given by*

$$\begin{aligned} \tilde{X}_{\tilde{\lambda}}(\tilde{x},\, \tilde{y}) &= \big(a\tilde{x} + b\tilde{y} + e\tilde{\lambda} + \tilde{R}_2^1, c\tilde{x} + d\tilde{y} + f\tilde{\lambda} + \tilde{R}_2^2\big) \\ \tilde{Y}_{\tilde{\lambda}}(\tilde{x},\, \tilde{y}) &= (0, \tilde{\mu}) \end{aligned} \tag{6}$$

*where $a,\, b,\, c,\, d,\, e,\, f,\, \tilde{\mu}$ are real constants and $\tilde{R}_2^1$ and $\tilde{R}_2^2$ are remainders with terms of order greater than or equal to $2$ in $\tilde{x}, \tilde{y}$ and $\tilde{\lambda}$, given by* (5)*. If $\varepsilon > 0$, then there exists a change of parameters so that the expression of the regularization of $\tilde{Z}_{\tilde{\lambda}}$ is family $Z_{\lambda,\varepsilon}$ given by*

$$Z_{\lambda,\varepsilon}(x, y) = \big(Z^1_{\lambda,\varepsilon}(x, y), Z^2_{\lambda,\varepsilon}(x, y)\big)$$

*where*

$$\begin{aligned} Z^1_{\lambda,\varepsilon}(x, y) &= \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+\varepsilon^2}}\right)\big(ax + by + e\lambda + R_2^1(x, y, \lambda)\big), \\ Z^2_{\lambda,\varepsilon}(x, y) &= \left(\frac{1}{2} - \frac{y}{2\sqrt{y^2+\varepsilon^2}}\right)\mu + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+\varepsilon^2}}\right)\big(cx + dy \\ &\quad + f\lambda + R_2^2(x, y, \lambda)\big). \end{aligned}$$

*Furthermore, the remainders $R_2^1$ and $R_2^2$ vanish when the parameter of regularization tends to zero.*

*Proof.* Let $\varepsilon > 0$ and $\tilde{Z}_{\tilde{\lambda}} = \big(\tilde{X}_{\tilde{\lambda}},\, \tilde{Y}_{\tilde{\lambda}}\big)$ be defined by (6). By (2) we have that the expression of the regularized family of vector fields is given by

$$Z_{\tilde{\lambda},\varepsilon}(\tilde{x},\, \tilde{y}) = \big(Z^1_{\tilde{\lambda},\varepsilon}(\tilde{x},\, \tilde{y}),\, Z^2_{\tilde{\lambda},\varepsilon}(\tilde{x},\, \tilde{y})\big)$$

where

$$
\begin{aligned}
Z^1_{\tilde{\lambda},\varepsilon}(\tilde{x},\tilde{y}) &= \big(1-\varphi_\varepsilon(F(\tilde{x},\tilde{y}))\big)\tilde{Y}^1_{\tilde{\lambda}}(\tilde{x},\tilde{y}) + \varphi_\varepsilon(F(\tilde{x},\tilde{y}))\tilde{X}^1_{\tilde{\lambda}}(\tilde{x},\tilde{y}) \\
&= \left(\frac{1}{2}+\frac{\tilde{y}}{2\sqrt{\tilde{y}^2+\varepsilon^2}}\right)\big(a\tilde{x}+b\tilde{y}+e\tilde{\lambda}+\tilde{R}^1_2\big)
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
Z^2_{\tilde{\lambda},\varepsilon}(\tilde{x},\tilde{y}) &= \big(1-\varphi_\varepsilon(F(\tilde{x},\tilde{y}))\big)\tilde{Y}^2_{\tilde{\lambda}}(\tilde{x},\tilde{y}) + \varphi_\varepsilon(F(\tilde{x},\tilde{y}))\tilde{X}^2_{\tilde{\lambda}}(\tilde{x},\tilde{y}) \\
&= \left(\frac{1}{2}-\frac{\tilde{y}}{2\sqrt{\tilde{y}^2+\varepsilon^2}}\right)\tilde{\mu} + \left(\frac{1}{2}+\frac{\tilde{y}}{2\sqrt{\tilde{y}^2+\varepsilon^2}}\right)\big(c\tilde{x}+d\tilde{y}+f\tilde{\lambda}+\tilde{R}^2_2\big).
\end{aligned}
$$

Now, we use the following change of parameters and variables

$$\tilde{x}=\varepsilon x,\quad \tilde{y}=\varepsilon y,\quad \tilde{\lambda}=\varepsilon\lambda \quad \text{e} \quad \tilde{\mu}=\varepsilon\mu$$

and in (7) we obtain

$$\varepsilon x' = \left(\frac{1}{2}+\frac{\varepsilon y}{2\sqrt{\varepsilon^2 y^2+\varepsilon^2}}\right)\big(a\varepsilon x+b\varepsilon y+e\varepsilon\lambda+R^1_2\big)$$

where

$$
\begin{aligned}
R^1_2 = R^1_2(\varepsilon x,\varepsilon y,\varepsilon\lambda) &= X^1_{2,0,0}(\varepsilon x,\varepsilon y,\varepsilon\lambda)\varepsilon^2x^2 + X^1_{0,2,0}(\varepsilon x,\varepsilon y,\varepsilon\lambda)\varepsilon^2y^2 \\
&+ X^1_{0,0,2}(\varepsilon x,\varepsilon y,\varepsilon\lambda)\varepsilon^2\lambda^2 + X^1_{1,1,0}(\varepsilon x,\varepsilon y,\varepsilon\lambda)\varepsilon^2xy + X^1_{1,0,1}(\varepsilon x,\varepsilon y,\varepsilon\lambda)\varepsilon^2x\lambda \\
&+ X^1_{0,1,1}(\varepsilon x,\varepsilon y,\varepsilon\lambda)\varepsilon^2y\lambda.
\end{aligned}
$$

So it is possible to cancel the parameter $\varepsilon$. In $R^1_2$ we have, after the simplification, the presence of the parameter $\varepsilon$ in all of its terms which make that this expression vanishes when $\varepsilon\to 0$. So we have that the first coordinate of the family of regularized vector fields, when $\varepsilon\to 0$, is

$$x' = \left(\frac{1}{2}+\frac{y}{2\sqrt{y^2+1}}\right)\big(ax+by+e\lambda\big).$$

By analogous arguments, we verify the expression for the second coordinate of $Z_{\lambda,\varepsilon}$. □

So the regularized family of vector fields, $Z_{\lambda,\varepsilon}$, has the following expression

$$x' = \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)(ax + by + e\lambda)$$
$$y' = \left(\frac{1}{2} - \frac{y}{2\sqrt{y^2+1}}\right)\mu + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)(cx + dy + f\lambda).$$

As a consequence of the proposition 3, we assume that the expression for the one parameter family of discontinuous vector field $Z_\lambda = (X_\lambda, Y_\lambda)$ with one singularity in the north set, for negative values of the parameter, that collides with the origin of the plane when $\lambda = 0,$ and with constant vector field in the south set is

$$\begin{aligned} X_\lambda(x,y) &= (ax + by + e\lambda,\, cx + dy + f\lambda) \\ Y_\lambda(x,y) &= (0,\, \mu) \end{aligned} \tag{8}$$

where $a,\, b,\, c,\, d,\, e,\, f$ are real constants, $\mu = \pm 1$ and $\lambda$ belongs to a small interval around the origin.

## 3 SADDLE AND NODE BOUNDARY BIFURCATION

In this section we present the bifurcations that occur when the singularity in the north set, of type saddle or node, collides with the origin when parameter vanishes. We start by defining the generic cases of a collision of a saddle in the north set with the discontinuity set.

As noted in Filippov (1998) or Kuznetsov et al. (2003), if the singularity of $X_\lambda$ is a saddle, then the respective bifurcation is represented by three generic cases characterized by the slope of the saddle zero isoclines. The main difference between our and their presentation is that we consider the discontinuity set fixed, whereas they consider it as a variable set.

Let us assume that $\lambda = 0$ and consider $m_u$ and $m_i$ be, respectively, the slope of the unstable set of the saddle for the vector field $X_0,$ and the slope of the saddle zero isocline. If $m_i < 0$ and $m_u > 0$ then we have the first case of saddle boundary bifurcation; if $m_i < m_u < 0$ we have the second case and, finally, to the third case we must have $m_u < m_i < 0.$ These are the characterizations that allow us to have the following results.

**Theorem 1.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one-parameter family of discontinuous vector fields on $M$ given by*

$$\begin{aligned} X_\lambda(x,y) &= (ax + by + e\lambda,\, cx + dy + f\lambda) \\ Y_\lambda(x,y) &= (0,\, 1) \end{aligned} \tag{9}$$

*where $a,\, b,\, c,\, d,\, e,\, f$ are real coefficients and $\lambda \in J$, where $J$ is a small interval around the origin. If $ad - bc < 0$, $ce - af > 0$, $a < 0$ and*

$$-\frac{c}{d} < 0 < \frac{d - a - \sqrt{(a-d)^2 + 4bc}}{2b},$$

*then $Z_\lambda$ is such that for small and negative values of the parameter $\lambda$ there exists, on north set, a saddle $s_\lambda$ which coexists with a stable node $n_\lambda$, of the Fillipov vector field, and an invisible tangent point $t_\lambda$. The unstable set of the saddle $s_\lambda$, for small and negative values of $\lambda$, intersects set $D$ in the point $u_\lambda$, and we have that $n_\lambda$ belongs to the complement of the interval $[t_\lambda,\, u_\lambda]$ on the sliding set. The points $n_\lambda,\, t_\lambda$ and $u_\lambda$ collide at the origin of the plane when the parameter vanishes. For small and positive values of the parameter, there is only a visible tangency at $t_\lambda$.*

*Proof.* The singularities of the family $X_\lambda$ are points $(x, y)$ on the plane where

$$\begin{aligned} x &= \frac{bf - de}{ad - bc}\lambda \\ y &= \frac{ce - af}{ad - bc}\lambda. \end{aligned}$$

These expressions are well defined since the singularity of $X_\lambda$ is a saddle.

Using the function $F(x,\, y) = y$ that defines the discontinuity set $D$, we have that

$$\begin{aligned} X_\lambda F &= (0,\, 1) \cdot (ax + by + e\lambda,\, cx + dy + f\lambda) = cx + dy + f\lambda \\ Y_\lambda F &= (0,\, 1) \cdot (0,\, 1) = 1 \end{aligned}$$

which implies that $(X_\lambda F)(Y_\lambda F) = cx + f\lambda$ on $D$.

So the sewing arc is the set of points in set $D$ where $cx + f\lambda > 0$. Let us suppose that $c \neq 0$, the point in set $D$, given by $x = -f\lambda/c$ is the point of invisible tangency of $X_\lambda$, i.e., we have that $t_\lambda = (-f\lambda/c, 0)$. Also, this is the transition point between the sewing and sliding sets. So the sewing set is characterized by $x > -\dfrac{f\lambda}{c}$ if $c > 0$, and $x < -\dfrac{f\lambda}{c}$ if $c < 0$. The sliding set satisfies $x < -\dfrac{f\lambda}{c}$ if $c > 0$, and $x > -\dfrac{f\lambda}{c}$ if $c < 0$.

By (1) the expression of the Filippov vector field, $F_{Z_\lambda}$, is

$$F_{Z_\lambda}(x, y) = \left(\frac{ax + e\lambda}{1 - cx - f\lambda}, 0\right).$$

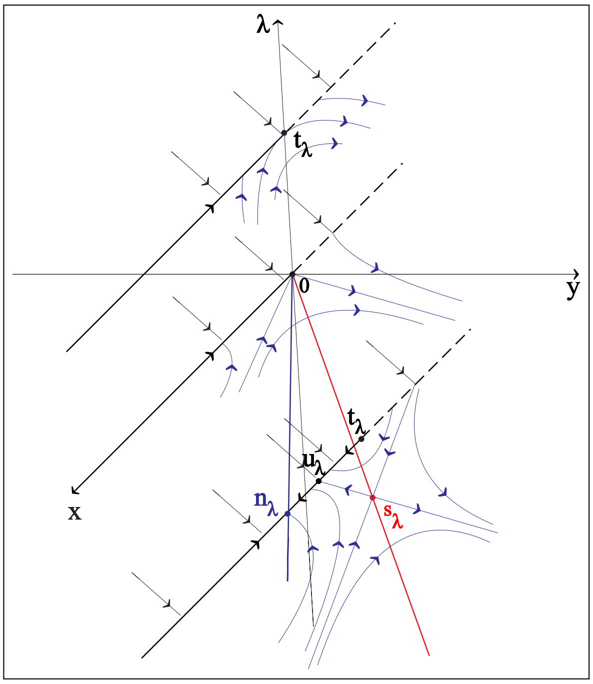
So the unique equilibrium of the Filippov vector field is the point $n_\lambda = (x, 0)$ where $x = -e\lambda/a$. If $a < 0$ and $\lambda < 0$ then we have that $n_\lambda$ belongs to the sliding set. Furthermore, $d\,(\det[X_\lambda, Y_\lambda]|_D)\,(n_\lambda) = a < 0$ implying that $n_\lambda$ is a stable node for negative values of the parameter $\lambda$.

The slope of the unstable set for the saddle when $\lambda = 0$ is given by $m_u = \dfrac{d - a - \sqrt{(a-d)^2 + 4bc}}{2b}$, which must be positive to this case, and the slope of the saddle zero isoclines is $m_i = -\dfrac{c}{d}$ and this must be negative.

For positive values of the parameter $\lambda$ there are no singularities for $Z_\lambda$. □

We say that $Z_\lambda = (X_\lambda, Y_\lambda)$ is of type $\mathcal{BS}_1$, if it is given by (9) and the coefficients satisfies the hypothesis of Theorem 1. The figure 2 presents the diagram of the bifurcation associated to a family of discontinuous vector fields of type $\mathcal{BS}_1$.

Figure 2 – Diagram of the case $\mathcal{BS}_1$




Source: the authors (2023)

**Theorem 2.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one parameter family of discontinuous vector fields on $M$ given by*

$$X_\lambda(x,y) = (ax + by + e\lambda,\ cx + dy + f\lambda)$$
$$Y_\lambda(x,y) = (0,\ 1)$$

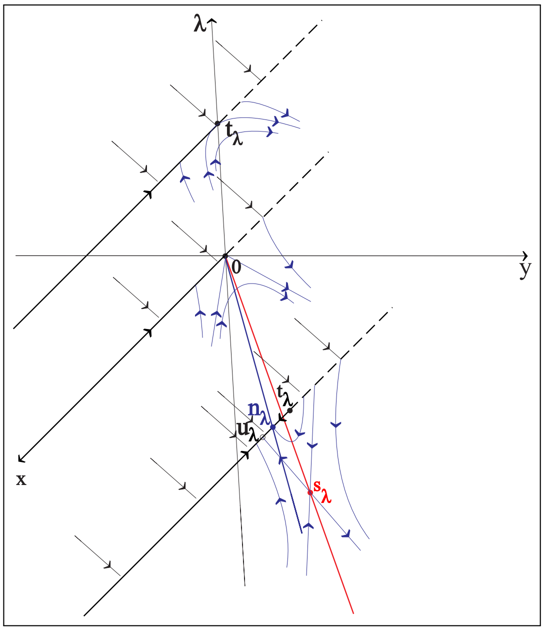
*where $a,\ b,\ c,\ d,\ e,\ f$ are real coefficients and $\lambda \in J$, where $J$ is a small interval around the origin. If $ad - bc < 0$, $ce - af > 0$, $a < 0$ and*

$$-\frac{c}{d} < \frac{d - a - \sqrt{(a-d)^2 + 4bc}}{2b} < 0,$$

*then $Z_\lambda$ is such that for small and negative values of the parameter $\lambda$ there exists, on north set, a saddle $s_\lambda$ which coexists with a stable node $n_\lambda$, of the Fillipov vector field, and an invisible tangent point $t_\lambda$. The unstable set of the saddle $s_\lambda$, for small and negative values of $\lambda$, intersects set $D$ in the point $u_\lambda$, and we have that $n_\lambda$ belongs to the interval $(t_\lambda,\ u_\lambda)$. The points $n_\lambda$, $t_\lambda$ and $u_\lambda$ collide at the origin of the plane when the parameter vanishes. For small and positive values of the parameter, there is only a visible tangency at $t_\lambda$.*

*Proof.* The proof of this theorem is based on the calculations made in the proof of the previous one. We need to make just one modification, which is $m_i < m_u < 0$. □

Figure 3 – Diagram of the case $\mathcal{BS}_2$




Source: the authors (2023)

We say that $Z_\lambda = (X_\lambda, Y_\lambda)$ is of type $\mathcal{S}_2$, if it is given by (9) and the coefficients satisfies the hypothesis of Theorem 2. Figure 3 presents the diagram of this case.

**Theorem 3.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one parameter family of discontinuous vector fields on $M$ given by*

$$X_\lambda(x,y) = (ax + by + e\lambda,\, cx + dy + f\lambda)$$
$$Y_\lambda(x,y) = (0,\, 1)$$

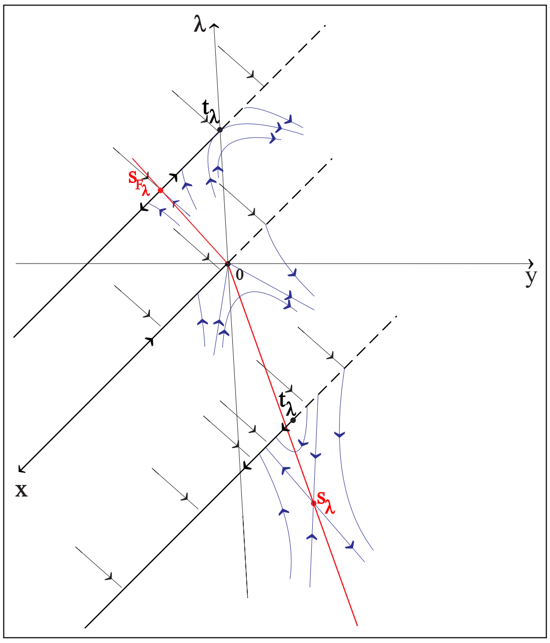
*where $a, b, c, d, e, f$ are real coefficients and $\lambda \in J$, where $J$ is a small interval around the origin. If $ad - bc < 0$, $ce - af > 0$, $a < 0$ and*

$$\frac{d - a - \sqrt{(a-d)^2 + 4bc}}{2b} < -\frac{c}{d} < 0,$$

*then $Z_\lambda$ is such that for small and negative values of the parameter $\lambda$ there exists, on north set, a saddle $s_\lambda$ which coexists with an invisible tangent point $t_\lambda$. These points collide at the origin of the plane when the parameter vanishes, and for small and positive values of $\lambda$ there is a saddle $s_{F\lambda}$ of the Fillipov vector field and a visible tangency at $t_\lambda$.*

*Proof.* The proof of this theorem is analogous to the previous theorems. □

Figure 4 – Diagram of the case $\mathcal{BS}_3$




Source: the authors (2023)

We say that $Z_\lambda = (X_\lambda, Y_\lambda)$ is of type $\mathcal{BS}_3,$ if it is given by (9) and the coefficients satisfies the hypothesis of Theorem 3.

The diagram of this case is in figure 4.

If the singularity of the family of vector fields $X_\lambda$ that collides with the origin is a stable node, we have two generic critical cases called $\mathcal{BN}_1$ and $\mathcal{BN}_2$.

($\mathcal{BN}_1$) We say that $Z_\lambda$ is a family of discontinuous vector fields of type $\mathcal{BN}_1$ if, for small and negative values of the parameter $\lambda$ there exists, on north set, a stable node $n_\lambda$ which coexists with a visible tangent point $t_\lambda$. These points collide at the origin of the plane when the parameter vanishes, and for small and positive values of $\lambda$ there is a stable node $n_\lambda$ of the Fillipov vector field and an invisible tangency at $t_\lambda$.

The figure 5(a) presents the diagram of this case.

($\mathcal{BN}_2$) We say that $Z_\lambda$ is a family of discontinuous vector fields of type $\mathcal{BN}_2$ if, for small and negative values of the parameter $\lambda$ there exists, on north set, a stable node $n_\lambda$ which coexists with a saddle point $s_\lambda$ of the Filippov vector field. These points collide at the origin of the plane when the parameter vanishes, and for small and positive values of $\lambda$ there is only one invisible tangency at $t_\lambda$.

The figure 5(b) presents the diagram of this case.

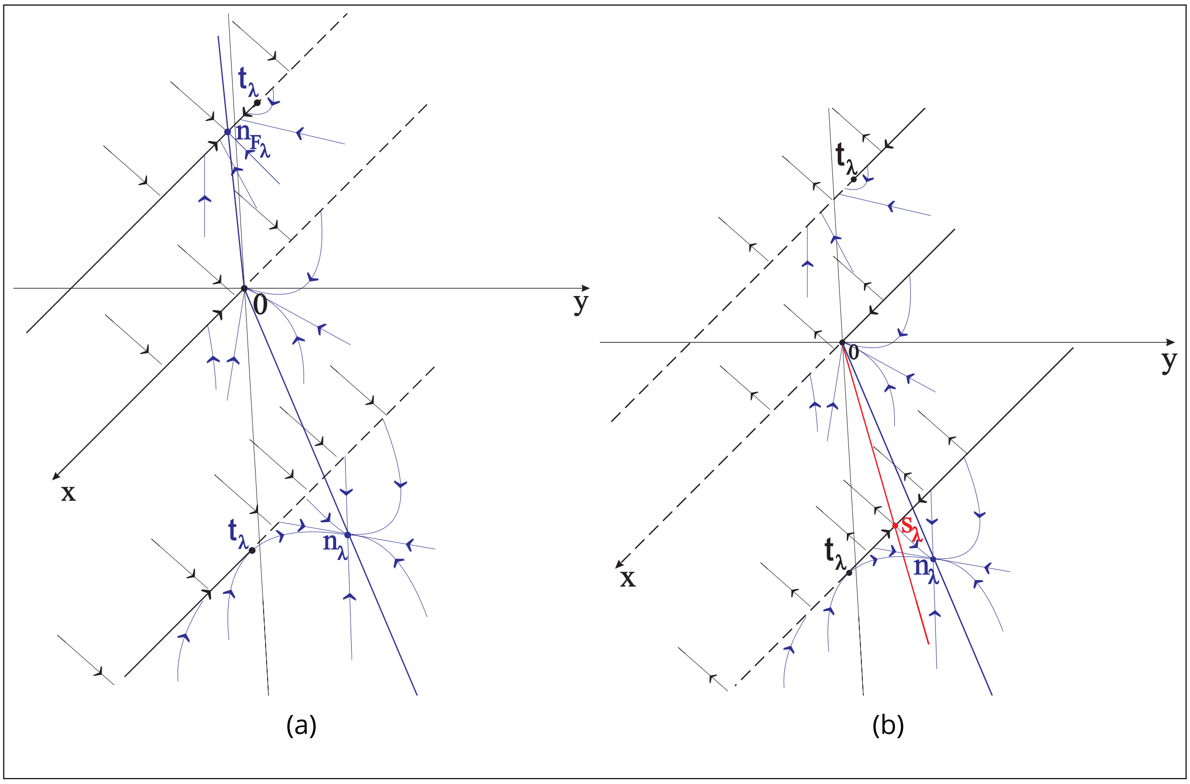
**Proposition 4.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a family of discontinuous vector field on $M$ given by* (8). *Let us suppose that $ce - af < 0$, $ad - bc > 0$ and $(a + d)^2 - 4(ad - bc) > 0$. Then the following relations characterizes each type of the node boundary bifurcations.*

*a) If $\mu = 1$, $a + d < 0$, $a < 0$ and $c < 0$ then $Z_\lambda$ is of type $\mathcal{BN}_1$.*

*b) If $\mu = -1$, $a + d < 0$, $a < 0$ and $c < 0$ then $Z_\lambda$ is of type $\mathcal{BN}_2$.*

*Proof.* In proof of the previous theorems we see that the sewing set is given by $\{x : cx + f\lambda > 0\}$, so this give us that $c < 0$ for items a) and b). Another fact is the presence of the singularity of the Filippov vector field and this lead us to the sign of $a$, i.e., $a < 0$ for booth items. By the definition of saddle, given in the introduction, we see that the singularity of the Filippov vector field is a saddle to the case b). □

Figure 5 – Diagrams of the cases $\mathcal{BN}_1$ and $\mathcal{BN}_2$




Caption: The (a) figure displays the Diagram of the case $\mathcal{BN}_1$ and (b) Diagram of the case $\mathcal{BN}_2$
Source: the authors (2023)

## 4 BASIC ANALYSIS OF THE REGULARIZED FAMILIES OF DISCONTINUOUS VECTOR FIELDS

If the family of discontinuous vector fields $Z_\lambda = \big(X_\lambda,\, Y_\lambda\big)$ is given by (8) with $\mu = 1$, the associated family of regularized vector fields, $Z_{\lambda,\varepsilon}$, is given by

$$\begin{aligned} x' &= \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)\big(ax + by + e\lambda\big),\\ y' &= \frac{1}{2} - \frac{y}{2\sqrt{y^2+1}} + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)\big(cx + dy + f\lambda\big). \end{aligned} \tag{10}$$

The singularities of $Z_{\lambda,\varepsilon}$, are points on the plane $(x, \lambda)$ so that

$$\begin{aligned} x &= \frac{(de - bf)y + 2ey(y - \sqrt{y^2+1}) + e}{af - ce}\\ \lambda &= \frac{(ad - bc)y + 2ay(y - \sqrt{y^2+1}) + a}{ce - af}. \end{aligned} \tag{11}$$

If we look at $\lambda$, given by (11), as a curve on the plane $(y,\lambda)$ then this curve gives us the exact quantity of singularities for each $\lambda$. The next two propositions give us these quantities for the different types of discontinuous vector fields presented in the previous section.

**Proposition 5.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a family of discontinuous vector fields given by*

$$x' = \big(ax + by + e\lambda,\, cx + dy + f\lambda\big)$$
$$y' = \big(0,\, 1\big).$$

a) *If $Z_\lambda$ is of type $\mathcal{BS}_1$ or $\mathcal{BS}_2$, then the regularized family of vector fields $Z_{\lambda,\varepsilon}$ given by* (10) *may have one, two or no singularities depending on the values of the parameter $\lambda$. The expression of the ordinate of this unique singularity is given by*

$$y^* = \frac{U^2 - 192a^2\delta^2 + \delta^4 - \delta^2 U}{24a\delta U} \tag{12}$$

*where $\delta = ad - bc$ and*

$$U = \sqrt[3]{\delta^2\big(576a^2\delta^2 + 3456a^4 - \delta^4 + 24\sqrt{3}a(\delta^2 + 8a^2)\sqrt{108a^2 + \delta^2}\big)}.$$

b) *If $Z_\lambda$ is of type $\mathcal{BS}_3$, then the regularized family of vector fields $Z_{\lambda,\varepsilon}$ given by* (10) *has one singularity.*

*Proof.* Differentiating $\lambda(y)$, given by (11), in relation to $y$ and simplifying it, we have that the critical point of this curve is the real root of

$$8a\delta y^3 + \delta^2 y^2 + 8a\delta y + \delta^2 - 4a^2 = 0. \tag{13}$$

This real root is the point $y^*$ given by (12).

It is easy to verify that, for positive values of $\dfrac{ad-bc}{a}$, this function has a unique maximum local point, and this happens when $a < 0$, i.e., when $Z_\lambda$ is of type $\mathcal{BS}_1$ or $\mathcal{BS}_2$. For negative values of the quantity $\dfrac{ad-bc}{a}$ this unique critical point is an inflection point and there is only one singularity for any $\lambda$, this happens when $Z_\lambda$ is of type $\mathcal{BS}_3$. □

**Proposition 6.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a family of discontinuous vector fields given by* (8)*.*

*a) If $Z_\lambda$ is of type $\mathcal{BN}_1$ then the regularized family of vector fields $Z_{\lambda,\varepsilon}$ given by* (10) *have one singularity.*

*b) If $Z_\lambda$ is of type $\mathcal{BN}_2$ then the regularized family of vector fields $Z_{\lambda,\varepsilon}$ is given by*

$$\begin{aligned} x' &= \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)\big(ax + by + e\lambda\big), \\ y' &= -\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}} + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)\big(cx + dy + f\lambda\big). \end{aligned} \tag{14}$$

*may have one, two or no singularities depending on the values of the parameter $\lambda$. The expression of the ordinate of this unique singularity given by*

$$y^* = \frac{U^2 - 192a^2\delta^2 + \delta^4 + \delta^2 U}{24a\delta U}$$

*where $\delta = ad - bc$ and*

$$U = \sqrt[3]{\delta^2\big(576a^2\delta^2 - 3456a^4 + \delta^4 + 24\sqrt{3}a(\delta^2 + 8a^2)\sqrt{108a^2 + \delta^2}\big)}$$

*Proof.* Item a) is treated similarly as the corresponding item of the Proposition 5.

To check item b) we must note that the curve of singularities of $Z_{\lambda,\varepsilon}$ on the plane $(x, \lambda)$ is

$$\lambda = \frac{(ad - bc)y - 2ay(y - \sqrt{y^2+1}) - a}{ce - af}.$$

This smooth curve have just one maximal point, and its ordinate is $y^*$ given in the statement b). □

To analyze the type of each singularity presented in the previous propositions, we will need to work with $A = (a_{ij})$, $1 \leq i,\ j \leq 2$, which is the Jacobian matrix of $Z_{\lambda,\varepsilon}$, i.e., $D(Z_{\lambda,\varepsilon})(x, y)$. So let us start when $\mu = 1$, we have that

$$\begin{aligned} a_{11} &= \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right) a \\ a_{12} &= \frac{ax + by + e\lambda}{2(y^2+1)^{3/2}} + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right) b \end{aligned}$$

$$a_{21} = \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right) c$$

$$a_{22} = \frac{cx + dy + f\lambda - 1}{2(y^2+1)^{3/2}} + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right) d.$$

The determinant of $A$ is

$$\begin{aligned} \det(A) = & \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)^2 (ad - bc) \\ & + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right) \left(\frac{(ad-bc)y + (af - ce)\lambda - a}{2(y^2+1)^{3/2}}\right) \end{aligned} \tag{15}$$

and the trace of this matrix is

$$\text{tr}(A) = \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)(a + d) + \frac{cx + dy + f\lambda - 1}{2(y^2+1)^{3/2}}. \tag{16}$$

Replacing the expressions of $x$ and $\lambda,$ given by (11), in (15) and (16) we obtain

$$\text{det}_s(A) = \frac{y + \sqrt{y^2+1}}{4(y^2+1)^2}\left(\left(\delta(y^2+1) + 2ay\right)\sqrt{y^2+1} + (\delta y - 2a)(y^2+1)\right) \tag{17}$$

$$\text{tr}_s(A) = \frac{1}{2(1+y^2)^{3/2}}\left(\sigma(y^2+1)\left(y + \sqrt{y^2+1}\right) - 2\left(y^2 + 1 - y\sqrt{y^2+1}\right)\right) \tag{18}$$

where $\delta = ad - bc$ and $\sigma = a + d.$

**Proposition 7.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one-parameter family of discontinuous vector fields given by* (8) *with $\mu = 1.$ Let $Z_{\lambda,\varepsilon}$ the regularized family of discontinuous vector fields given by* (10)*, where the curve of its singularities is*

$$\lambda(y) = \frac{(ad-bc)y + 2ay(y - \sqrt{y^2+1}) + a}{ce - af}.$$

*Then, the following statements are valid:*

*a) The determinant of the Jacobian matrix vanish in the point $(y^*, \lambda(y^*))$ where $y^* = \dfrac{U^2 - 192a^2\delta^2 + \delta^4 - \delta^2 U}{24a\delta U}$, for $\delta = ad - bc$ and*

$$U = \sqrt[3]{\delta^2\left(576a^2\delta^2 + 3456a^4 - \delta^4 + 24\sqrt{3}a(\delta^2 + 8a^2)\sqrt{108a^2 + \delta^2}\right)}.$$

*b) If $Z_\lambda$ is of type $\mathcal{BS}_1$ or $\mathcal{BS}_2$, then there is a neighborhood $I$ of $y^*$ such that this determinant is negative if $y \in I$ and $y > y^*$, and is positive if $y \in I$ and $y < y^*$.*

*Proof.* a)We have already obtained the point $y^*$ in (12). On the other hand, simplifying (17) we obtain that the determinant of Jacobian vanishes exactly on the root of (13), i.e., $\det_s(A)(y^*) = 0$. All other points nearby $y^*$ we have that the determinant is not null.

To check item b) we have that

$$\frac{d\det_s(A)}{dy}(y^*) = \frac{y^* + \sqrt{(y^*)^2+1}}{2((y^*)^2+1)^{5/2}}\Big( -3a(y^*)^2 + (3ay^*+\delta)\sqrt{(y^*)^2+1}\Big). \tag{19}$$

Considering that $y^*$ is the critical point of $\lambda(y)$, we have that $-4a(y^*)^2 + (4ay^*+\delta)\sqrt{(y^*)^2+1} = 2a$. Replacing this value in (19) we obtain that

$$\frac{d\det_s(A)}{dy}(y^*) = \frac{y^* + \sqrt{(y^*)^2+1}}{2((y^*)^2+1)^{5/2}} a\Big(2 + (y^*)^2 - y^*\sqrt{(y^*)^2+1}\Big)$$

which depends on the sign of $a$. Since, for the cases $\mathcal{BS}_1$ and $\mathcal{BS}_2$ we have $a < 0$, and by the continuity of the determinant, there exists a neighborhood $I$ of $y^*$ such that for $y > y^*$ the determinant of the Jacobian matrix is negative, and the determinant is positive if $y < y^*$. □

**Proposition 8.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one-parameter family of discontinuous vector fields of type $\mathcal{BN}_2$. Let $Z_{\lambda,\varepsilon}$ be the regularized family of discontinuous vector fields given by* (14)*, where the curve of its singularities is*

$$\lambda(y) = \frac{(ad-bc)y - 2ay(y-\sqrt{y^2+1}) - a}{ce - af}.$$

*Then there is a neighborhood $I$ of $y^*$ such that this determinant is negative if $y \in I$ and $y > y^*$, and is positive if $y \in I$ and $y < y^*$.*

The proof of this proposition is a slight modification on the proof of Proposition 7 and we will omit it.

We end this section with a general result related to the type of singularities of discontinuous vector fields from the previous section.

**Proposition 9.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one-parameter family of discontinuous vector fields given by* (8)*. Let $Z_{\lambda,\varepsilon}$ the regularized family of the discontinuous vector fields* (8)*. Then, the following statements are true:*

*a) If $Z_\lambda$ is a family of discontinuous vector fields of type $\mathcal{BS}_1$ or $\mathcal{BS}_2$, then for $y < y^*$ the singularity is a node, and is a saddle if $y > y^*$, where $y^*$ is given in the item a) of Proposition 5.*

*b) If $Z_\lambda$ is a family of discontinuous vector fields of type $\mathcal{BS}_3$, then for any value of $y$ the singularity is a saddle.*

*c) If $Z_\lambda$ is a family of discontinuous vector fields of type $\mathcal{BN}_1$, then for any value of $y$ the singularity is a node.*

*d) If $Z_\lambda$ is a family of discontinuous vector fields of type $\mathcal{BN}_2$, then for $y > y^*$ the singularity is a node, and for $y < y^*$ the singularity is a saddle, where $y^*$ is given in item b) of Proposition 6.*

*Proof.* The proof of a) follows directly from Proposition 7, and d) follows from Proposition 8.

Items b) and c) follow directly from Proposition 1.

□

## 5 REGULARIZED SADDLE BOUNDARY BIFURCATION

In this section, we will explain the saddle boundary bifurcations in terms of the classical smooth bifurcations. For the cases $\mathcal{BS}_1$ and $\mathcal{BS}_2$, we will prove that when the parameter is $\lambda^*$, related to the value $y^*$ given by (12), there occurs a saddle-node bifurcation. For the case $\mathcal{BS}_3$, we have just a saddle that moves from $S$ to $N$. To obtain the saddle-node bifurcation, we use the following theorem, see Sotomayor (1973) or Guckenheimer & Holmes (1983).

**Theorem 4** (Sotomayor)**.** *Let $\dot{x} = G(x, \lambda)$ be a one-parameter family of differential equations in $\mathbb{R}^2$, where $\lambda \in \mathbb{R}$ is the parameter. When $\lambda = \lambda_0$, we assume that there is a singularity $p_0 = (x_0, y_0)$ which satisfies:*

*(SN1) the jacobian matrix of $G$ applied in $(p_0, \lambda_0)$ has only one zero eigenvalue $\mu_0$ with, respectively, right and left eigenvectors $v$ and $w$;*

*(SN2) the following inequality is valid*

$$\left\langle w, \frac{d}{d\lambda}G(p_0, \lambda_0)\right\rangle \neq 0$$

*where $<,>$ is the usual inner product of the plane;*

*(SN3) is valid*

$$\langle w, D_x^2 G(p_0, \lambda_0)(v, v)\rangle \neq 0.$$

*Then there is a smooth curve of singularities in $\mathbb{R}^2 \times \mathbb{R}$ passing by $(p_0, \lambda_0)$, and tangent to the hyperplane $\mathbb{R}^2 \times \{\lambda_0\}$. Depending on the signs of the expressions in (SN2) and (SN3), there are no singularities close to $(p_0, \lambda_0)$ when $\lambda < \lambda_0$ $(\lambda > \lambda_0)$ and two singularities close to $(p_0, \lambda_0)$ to each value of the parameter $\lambda > \lambda_0$ $(\lambda < \lambda_0)$. Both the singularities of $\dot{x} = G(x, \lambda)$ that are close to $(p_0, \lambda_0)$ are hyperbolic.*

Let us apply this theorem to our cases.

**Theorem 5.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one-parameter family of discontinuous vector fields of type $\mathcal{BS}_1$ or $\mathcal{BS}_2$. Let $Z_{\lambda,\varepsilon}$ be the regularized family of discontinuous vector fields given by* (10)*. Then $Z_{\lambda,\varepsilon}$ has a saddle-node point given by the expression $p_{SN} = \big(x_{SN}, y_{SN}, \lambda_{SN}\big)$ where*

$$y_{SN} = \frac{U^2 - 192a^2\delta^2 + \delta^4 - \delta^2 U}{24a\delta U},$$

*for $\delta = ad - bc$ and*

$$U = \sqrt[3]{\delta^2\big(576a^2\delta^2 + 3456a^4 - \delta^4 + 24\sqrt{3}a(\delta^2 + 8a^2)\sqrt{108a^2 + \delta^2}\big)}.$$

*The expressions of $x_{SN}$ and $\lambda_{SN}$ are given, respectively, by* (11)*, changing $y$ into $y_{SN}$.*

*Proof.* Let $Z_\lambda = (X_\lambda, Y_\lambda)$ a one-parameter family of discontinuous vector fields of type $\mathcal{BS}_1$ or $\mathcal{BS}_2$, and let $Z_{\lambda,\varepsilon}$ be the regularized family of discontinuous vector fields given by (10). Solving $\det_s(A) = 0$, where $\det_s$ is (17), related to the variable $y$ we obtain the point of coordinates $p^* = (x^*, y^*)$ and the parameter $\lambda^*$ that satisfie (SN1) from the Theorem 4. Let us check that $(x^*, y^*, \lambda^*)$ is a saddle-node point from the previous Theorem.

The right eigenvector associated to the eigenvalue $\mu_1 = 0$ is the non-zero vector $v = (v_1, v_2)$ which satisfy $A(p^*, \lambda^*)v = \mu_1 v = 0$. So

$$v = \left(1, \frac{-a\left(\sqrt{1+(y^*)^2}+y^*\right)(1+(y^*)^2)}{ax^*+b(y^*)^3+2by^*+e\lambda^*+b(1+(y^*)^2)^{3/2}}\right) \tag{20}$$

similarly the left eigenvector associated to the eigenvalue $\mu_1 = 0$ is the non-zero solution of $w^T A(p^*, \lambda^*) = 0$ which has the following expression

$$w = \left(1, -\frac{a}{c}\right).$$

This is the step (SN1) of Theorem 4.

Now, (SN2) is equivalent to show that the following inner product does not vanish

$$p_1 = \left\langle w, \frac{d}{d\lambda} G(p^*, \lambda^*) \right\rangle,$$

where $G(x, y, \lambda)$ is the regularized family of discontinuous vector fields, i. e.,

$$G(x, y, \lambda) = Z_{\lambda,\varepsilon}(x, y, \lambda) = \left(Z^1_{\lambda,\varepsilon}(x, y), Z^2_{\lambda,\varepsilon}(x, y)\right).$$

For this we have that

$$\frac{d}{d\lambda} Z_{\lambda,\varepsilon}(x^*, y^*, \lambda^*) = \left(\left(\frac{1}{2} + \frac{y^*}{2\sqrt{(y^*)^2+1}}\right)e, \left(\frac{1}{2} + \frac{y^*}{2\sqrt{(y^*)^2+1}}\right)f\right)$$

and multiplying by $w$ we have that

$$p_1 = \left(\frac{1}{2} + \frac{y^*}{2\sqrt{(y^*)^2+1}}\right)\left(\frac{ce-af}{c}\right).$$

Which is non-zero by hypothesis, and this finishes the verifying of (SN2).

The final part is to prove that $p_2 \neq 0$ where

$$p_2 = \left\langle w, D^2_x Z_{\lambda,\varepsilon}(p^*, \lambda^*)(v, v)\right\rangle.$$

For this, we need to calculate $D^2_x Z_{\lambda,\varepsilon}(p^*, \lambda^*)(v, v)$, and we will use the following notation $(x, y) = (x_1, x_2)$. So

$$D_x^2 Z_{\lambda,\varepsilon}(p^*,\lambda^*)(v,v) = \left( \sum_{i,j=1}^{2} \frac{\partial^2 Z^1_{\lambda,\varepsilon}}{\partial x_i \partial x_j}(p^*,\lambda^*) v_j v_i, \sum_{i,j=1}^{2} \frac{\partial^2 Z^2_{\lambda,\varepsilon}}{\partial x_i \partial x_j}(p^*,\lambda^*) v_j v_i \right).$$

Calculating the first component of this derivative, we have that

$$\frac{\partial^2 Z^1_{\lambda,\varepsilon}}{\partial x_1^2}(p^*,\lambda^*)v_1^2 + 2\frac{\partial^2 Z^1_{\lambda,\varepsilon}}{\partial x_1 \partial x_2}(p^*,\lambda^*)v_1 v_2 + \frac{\partial^2 Z^1_{\lambda,\varepsilon}}{\partial x_2^2}(p^*,\lambda^*)v_2^2$$
$$= \frac{a v_1 v_2}{\big((y^*)^2+1\big)^{3/2}} + \frac{y^*(-3ax^* - 3e\lambda^* - by^*) + 2b}{2\big((y^*)^2+1\big)^{5/2}} v_2^2,$$

the second component is

$$\frac{\partial^2 Z_2}{\partial x_1^2}(p^*,\lambda^*)v_1^2 + 2\frac{\partial^2 Z_2}{\partial x_1 \partial x_2}(p^*,\lambda^*)v_1 v_2 + \frac{\partial^2 Z_2}{\partial x_2^2}(p^*,\lambda^*)v_2^2$$
$$= \frac{c v_1 v_2}{\big((y^*)^2+1\big)^{3/2}} + \frac{y^*(3 - 3cx^* - dy^* - 3f\lambda^*) + 2d}{2\big((y^*)^2+1\big)^{5/2}} v_2^2.$$

So the expression of $p_2$ is

$$p_2 = \frac{v_2^2\big(\delta(y^*)^2 - 3ay^* - 3\nu y^*\lambda^* - 2\delta\big)}{2c\big((y^*)^2+1\big)^{5/2}}$$

where $\delta = ad - bc$ and $\nu = ce - af$. From the expression of $v$, given by (20), we have that $v_2 \neq 0$. Then, $p_2 = 0$ if $\delta(y^*)^2 - 3ay^* - 3\nu y^*\lambda^* - 2\delta = 0$. Replacing $\lambda^*$ by (11) and simplifying we have that $p_2 = 0$ if, and only if,

$$-3a(y^*)^3 - \delta(y^*)^2 + 3a(y^*)^2\sqrt{1+(y^*)^2} - 3ay^* - \delta = 0. \tag{21}$$

Comparing the expressions of (13), where one root is $y^*$, to (21), we notice that they do not vanish at the same point. And this finishes the proof of (SN3).

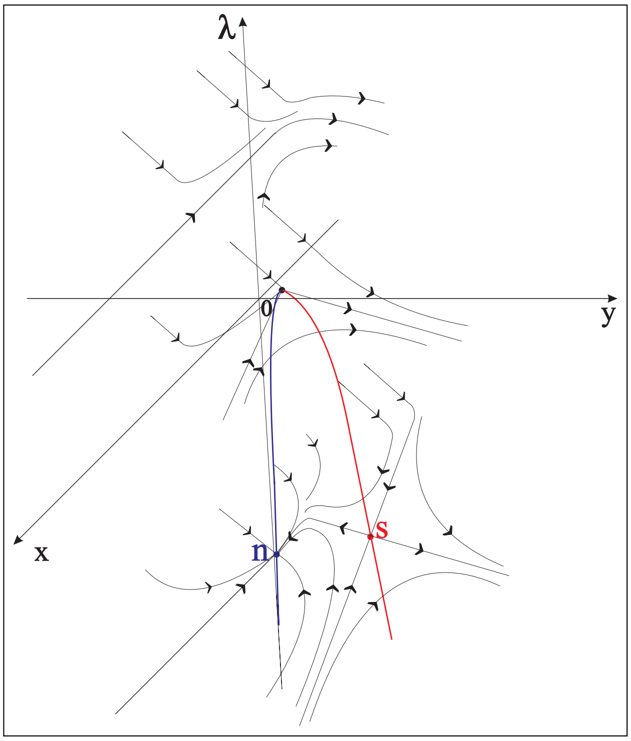
Then, the point $(x_{SN}, y_{SN}, \lambda_{SN}) = (x^*, y^*, \lambda^*)$ is a saddle-node point for family $Z_{\lambda,\varepsilon}$.

□

The figure 6 represent the diagram of bifurcation of a regularized family of discontinuous vector fields with a saddle-node bifurcation. Note that, with a small modification, this diagram is useful to explain the bifurcation that occur in $\mathcal{BS}_1$ and $\mathcal{BS}_2$.

The third case is simpler because it is a direct application of item b) of Proposition 5 and item b) of Proposition 9. In this case, the regularized family of discontinuous vector fields of type $\mathcal{BS}_3$ has a unique saddle for different values of parameter $\lambda$. So does not occur a bifurcation in the classic sense for this case.

Figure 6 – Diagram of bifurcation containing a saddle-node bifurcation




Source: the authors (2023)

## 6 REGULARIZED NODE BOUNDARY BIFURCATION

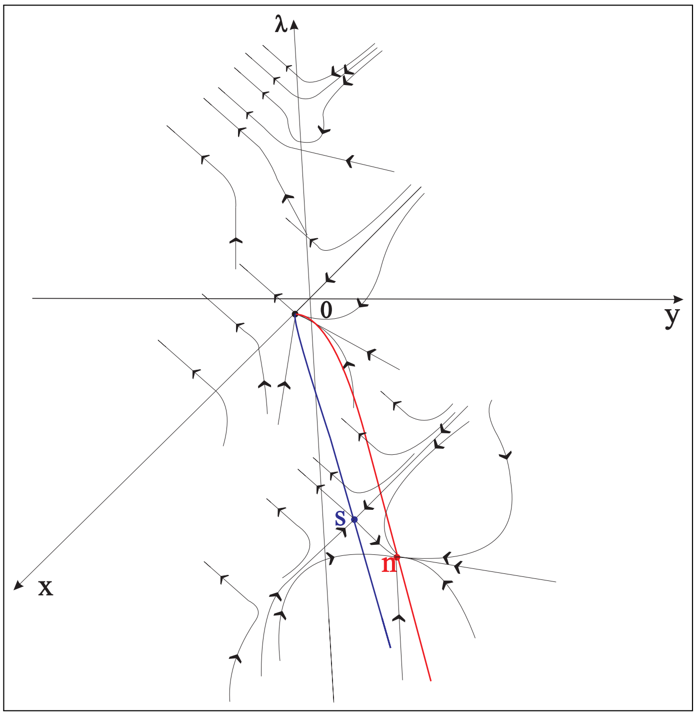
In the regularized family of vector fields associated to a discontinuous vector field of type $\mathcal{BN}_1$, we have the appearance of a stable node for any small values of the parameter. This is a direct consequence of Proposition 9 and the expression of the trace of the Jacobian matrix given by (18), which is negative for all values of the parameter. So in this case, we do not have a bifurcation.

Now, regarding a discontinuous family of vector fields of type $\mathcal{BN}_2$, its regularization is such that there is a saddle-node bifurcation.

**Theorem 6.** *Let $Z_\lambda = (X_\lambda, Y_\lambda)$ be a one-parameter family of discontinuous vector fields given by* (8)*. Let $Z_{\lambda,\varepsilon}$ be the regularized family of discontinuous vector fields given by* (10)*. If $Z_\lambda$ is of type $\mathcal{BN}_2$ then $Z_{\lambda,\varepsilon}$ has a saddle-node point given by the expression $p_{SN} = (x_{SN}, y_{SN}, \lambda_{SN})$ where $y_{SN}$ is given by $y^*$ in item b) of Proposition* (6)*.*

The proofs presented for the saddle boundary case can be easily adapted to this case. The diagram of bifurcation for the family of regularization associated to a discontinuous family of vector fields of type $\mathcal{BN}_2$ is presented in figure 7.

Figure 7 – Diagram of bifurcation of a node




Source: the authors (2023)

# 7 CONCLUSION

The regularized method of discontinuous vector fields is an important tool to understand discontinous bifurcations in classical sense. In this work we present two discontinuous bifurcations with codimension one, obtained by the colision of a saddle and a node with the discontinuity set. Professor Jorge Sotomayor conjectured that all discontinuous codimension one bifurcation given by Filippov (1998) and Kuznetsov et al. (2003), can be explained by classical bifurcations, using the regularization method. Our work is a small contribution to the solution of this problem.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