CIÊNCIA e NATURA

ISSN 2179-460X
OPEN ACCESS

Ci. e Nat., Santa Maria, v. 46, e75968, 2024• https://doi.org/10.5902/2179460X75968
Submitted: 04/24/23 • Approved: 11/10/23 • Published: 12/19/24

Mathematics

Trigonometric solutions that relate geometrical quantities of the triangle and the inscribed circle

Soluções trigonométricas que relacionam grandezas geométricas do triângulo e o círculo inscrito

Radael de Souza Parolin [I], **Alisson Darós Santos** [I]

[I]Universidade Federal do Pampa, RS, Brazil

ABSTRACT

This paper is concerned with the relations between a triangle and its inscribed circle. We obtained a specific class of trigonometric solutions that determine the area of the triangle from the radius of the inscribed circle and from two fixed sides of the triangle. To prove our results, we use trigonometric relations from Euclidean geometry, Viète formulas for roots of polynomial functions, Lagrange multipliers and Differential Calculus in one and two variables. In addition, we explain the behavior of these solutions through computer simulations including the intervals of existence and specific numerical cases. Moreover, we describe the relations between the triangle area and the inscribed circle area.

Keywords: Existence; Geometrical quantities; Polynomial equations; Trigonometric solutions

RESUMO

Este artigo trata das relações entre um triângulo e seu círculo inscrito. Obtivemos uma classe específica de soluções trigonométricas que determinam a área do triângulo a partir do raio do círculo inscrito e de dois lados fixos do triângulo. Para comprovar nossos resultados, utilizamos relações trigonométricas da geometria euclidiana, fórmulas de Viète para raízes de funções polinomiais, multiplicadores de Lagrange e cálculo diferencial em uma e duas variáveis. Além disso, explicamos o comportamento dessas soluções por meio de simulações computacionais incluindo os intervalos de existência e casos numéricos específicos. Ademais, descrevemos as relações entre a área do triângulo e a área do círculo inscrito.

Palavras-chave: Existência; Grandezas geométricas; Equações polinomiais; Soluções trigonométricas

1 INTRODUCTION

Historically, the development of mathematics has occurred in a solid way by relating different concepts of this science and also concrete situations of humanity. Geometry is closely linked to Differential and Integral Calculus, from its axiomatic origin with Euclid, through the Archimedes Exhaustion Method to the formalization of Calculus with Newton and Leibniz. This research is interested in analyzing the existence of relations between geometric quantities of the triangle and its inscribed circle, comparing from the equation

$$A = s \cdot r, \tag{1}$$

the area of the triangle (A), the semiperimeter (s) and the radius (r) of these geometric shapes. Note that equation (1), although it seems quite simple, hides important relations of geometry and with the use of the well-known Heron's formula (Oliver (1993))

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \tag{2}$$

where s is the semiperimeter $\frac{a+b+c}{2}$, result in classes of non-trivial trigonometric solutions like in (16), (17) and (18). Robbins (1994) uses generalizations of the equations (1) and (2) to present formulas of the areas of the pentagon and hexagon inscribed in a circle in terms of their sides. Maley et al. (2005) investigated the "generalized Heron polynomial" that relates the squared area of an n-gon inscribed in a circle to the squares of its side lengths. Generalizing to any n-gon, with odd n, Buck and Siddon (2012) present the area of polygons with inscribed circles. Nunes (2013), on the other hand, presents a representation for the ratio between the areas of the inscribed circle in a regular n-sided polygon. Considering the inscribed and escribed circles of a right triangle, Hansen (2003) addresses three theorems that present equations that relate the four radii of the circles to each other and also to the sides of the triangle. Still, this study presents an equation for the area of the equilateral triangle dependent of the four mentioned radii. Evans (1874) already presented geometric relations of the triangle and its six circles: the circumscribed, the inscribed, the nine-point and the three escribed. Brooks and Waksman (1987) bring the first eigenvalue of a scalene triangle which relates their area and perimeter.

Therefore, starting from equations (1) and (2), and replacing the semiperimeter of the triangle of sides $a, b, c \in \mathbb{R}^+$, with a and b fixed, we see that c must satisfy the polynomial equation

$$F(c) = 0,$$

where $F(x) = x^3 + d_2x^2 + d_1x + d_0$, with $d_2 = -(a+b)$, $d_1 = 4r^2 - (a-b)^2$ and $d_0 = (a+b)[4r^2+(a-b)^2]$, so that by the Cardano-Tartaglia formula (Cardano (1993); Lima (1987); Tartaglia (1554)), the polynomial function F has real roots. Considering the geometry of the problem, the intervals of existence of these roots are analyzed, highlighting the positive solutions.

In general, the solutions obtained for equation (1) have the form

$$A = \mu + \eta \cos(K \arccos(\theta) + L) \tag{3}$$

where $\mu = \mu(r, a, b)$, $\eta = \eta(r, a, b)$ and $\theta = \theta(r, a, b)$.

2 EXISTENCE OF A NONTRIVIAL FAMILY OF COSINE SOLUTIONS IN A TRIANGLE WITH FIXED PARAMETERS a AND b

In this section, we will establish the existence of a nontrivial solutions family that relate the area of an arbitrary triangle (A), with sides a, b and c, with the radius (r) of its inscribed circle. For this, we fixed a and b and keep c and r variable.

Relating equations (1) and (2) and replacing the semiperimeter $s = \frac{a+b+c}{2}$, we obtain

$$r = \frac{1}{2}\sqrt{\frac{-c^3 + (a+b)c^2 + (a-b)^2c - (a+b)(a-b)^2}{a+b+c}}. \tag{4}$$

Particularly, for the case of the right triangle with hypotenuse in c, equation (4) is given by

$$r = \frac{a+b-c}{2}. \tag{5}$$

From (4) we can see that for the existence of the triangle, $c \in (|a-b|, a+b)$. In order to obtain the family of solutions (3), we express c depending of r writing the cubic polynomial equation

$$F(c) = c^3 - (a+b)c^2 + [4r^2 - (a-b)^2]c + (a+b)[4r^2 + (a-b)^2] = 0, \quad (6)$$

for $F(x) = x^3 + d_2x^2 + d_1x + d_0$ with $d_2 = -(a+b)$, $d_1 = 4r^2 - (a-b)^2$ and $d_0 = (a+b)[4r^2 + (a-b)^2]$. From the Cardano-Tartaglia method, we obtain the discriminant

$$\Delta_1 = \frac{64}{27}\left[r^6 + (2a^2 + 7ab + 2b^2)r^4 + (a^4 - a^3b - a^2b^2 - ab^3 + b^4)r^2 - \frac{1}{4}a^2b^2(a-b)^2\right]. \quad (7)$$

Analyzing the sign of this discriminant Δ_1, we consider the polynomial $p_1(r) = r^6 + (2a^2 + 7ab + 2b^2)r^4 + (a^4 - a^3b - a^2b^2 - ab^3 + b^4)r^2 - \frac{1}{4}a^2b^2(a-b)^2$ and the change of variable $r^2 = x$, to obtain

$$p_2(x) = x^3 + (2a^2 + 7ab + 2b^2)x^2 + (a^4 - a^3b - a^2b^2 - ab^3 + b^4)x - \frac{1}{4}a^2b^2(a-b)^2.$$

Lemma 1: For $a, b > 0$, the polinomial function $p_2(x)$ has three real and distinct roots.
proof. Applying the Cardano-Tartaglia method to $p_2(x)$, we can write

$$\Delta_2 = -\frac{1}{1728}ab(a+b)^4(8a^2 - 11ab + 8b^2)^3.$$

Note that $\Delta_2 < 0$ for any $a > 0$ and $b > 0$ if, and only if, $8a^2 - 11ab + 8b^2 > 0$. As $(a^2 + b^2) \geq 2ab$, it follows that $8a^2 - 11ab + 8b^2 = 8(a^2 + b^2) - 11ab \geq 8(2ab) - 11ab = 5ab \geq 0$.■

Thus, we consider α, β and γ three real and distinct roots of $p_2(x)$ such that $\alpha < \beta < \gamma$.

Lemma 2: For $a, b > 0$, consider $\alpha \equiv \alpha(a, b) \in (I_0, I_1)$ such that

$$\begin{cases} I_0 = \dfrac{-2a^2 - 7ab - 2b^2 - 2\sqrt{a^4 + 31a^3b + 60a^2b^2 + 28ab^3 + 3ab^2 + b^4}}{3} \\ I_1 = \dfrac{-2a^2 - 7ab - 2b^2 + 2\sqrt{a^4 + 31a^3b + 60a^2b^2 + 28ab^3 + 3ab^2 + b^4}}{3}. \end{cases} \quad (8)$$

Then, we can write $\beta \equiv \beta(a,b)$ and $\gamma \equiv \gamma(a,b)$, such that

$$\beta = \frac{-\alpha - 2a^2 - 7ab - 2b^2 - \sqrt{\nu(\alpha,a,b)}}{2} \tag{9}$$

and

$$\gamma = \frac{-\alpha - 2a^2 - 7ab - 2b^2 + \sqrt{\nu(\alpha,a,b)}}{2}, \tag{10}$$

where $\nu(\alpha,a,b) = -3\alpha^2 - (4a^2 + 14ab + 4b^2)\alpha + 32a^3b + 61a^2b^2 + 28ab^3 + 4ab^2$.

proof. Initially, we will obtain the equation of an ellipse Σ depending on α and β, using the formulas of Viète (Girard (1884); Viète (1646)) for the roots of the polynomial of degree three $p_2(x)$, given by

$$\begin{cases} \alpha + \beta + \gamma = -2a^2 - 7ab - 2b^2 \\ \alpha\beta + \alpha\gamma + \beta\gamma = a^4 - a^3b - a^2b^2 - ab^3 + b^4 \\ \alpha\beta\gamma = \dfrac{1}{4}a^2b^2(a-b)^2. \end{cases}$$

So,

$$\begin{cases} \gamma = -2a^2 - 7ab - 2b^2 - \alpha - \beta \\ \gamma(\alpha+\beta) = a^4 - a^3b - a^2b^2 - ab^3 + b^4 - \alpha\beta \end{cases}$$

$$\Rightarrow \gamma = -2a^2 - 7ab - 2b^2 - \alpha - \beta = \frac{a^4 - a^3b - a^2b^2 - ab^3 + b^4 - \alpha\beta}{\alpha+\beta},$$

that, written equivalently as

$$\alpha^2 + \alpha\beta + \beta^2 + (2a^2 + 7ab + 2b^2)\alpha + (2a^2 + 7ab + 2b^2)\beta + a^4 - a^3b - a^2b^2 - ab^3 + b^4 = 0, \tag{11}$$

corresponds to the equation of an ellipse Σ in α and β. Here we consider $\alpha + \beta \neq 0$ which is verify in the sequence of paper.

Now we determine the variation intervals of α and β in the ellipse Σ. For this, consider again the equation of the ellipse Σ given by (11), like

$$\beta^2 + (\alpha + 2a^2 + 7ab + 2b^2)\beta + [\alpha^2 + (2a^2 + 7ab + 2b^2)\alpha + a^4 - a^3b - a^2b^2 - ab^3 + b^4] = 0. \tag{12}$$

This equation has a solution in β if, and only if, its discriminant Δ_3 is nonnegative. But,

$$\Delta_3 = -3\alpha^2 - (4a^2 + 14ab + 4b^2)\alpha + 32a^3b + 61a^2b^2 + 28ab^3 + 4ab^2,$$

then to obtain $\Delta_3 \geq 0$, we will study the second degree equation in α

$$3\alpha^2 + (4a^2 + 14ab + 4b^2)\alpha - 32a^3b - 61a^2b^2 - 28ab^3 - 4ab^2 = 0. \tag{13}$$

So, from the discriminant Δ_4 of the second degree equation (13), we see that for $a, b > 0$

$$\Delta_4 = 16(a^4 + 31a^3b + 60a^2b^2 + 28ab^3 + 3ab^2 + b^4) > 0.$$

Thus, the equation (13) has two real and distinct roots $I_0 < I_1$ as in (8). Consequently, for $\alpha \in (I_0, I_1)$ we have $\Delta_3 > 0$ and the equation (12) has solution so we can write (9). Also, as $\gamma = -2a^2 - 7ab - 2b^2 - \alpha - \beta$, it follows (10). ■

In order to adjust the existence intervals for the roots α, β and γ we will use the Lagrange multiplier method for the function $M(\alpha^*, \beta^*) \doteq \alpha^* + \beta^*$, subject to equality constraints (12), for $\alpha^* \in (I_0, I_1)$ and $\beta^* \equiv \beta^*(a, b)$ as in Lemma 2. Thus, denoting $N(\alpha^*, \beta^*) = (\beta^*)^2 + (\alpha^* + 2a^2 + 7ab + 2b^2)\beta^* + [(\alpha^*)^2 + (2a^2 + 7ab + 2b^2)\alpha^* + a^4 - a^3b - a^2b^2 - ab^3 + b^4]$, we have that

$$\nabla M(\alpha^*, \beta^*) = \lambda \nabla N(\alpha^*, \beta^*) \text{ e } N(\alpha^*, \beta^*) = 0$$

has a solution given by

$$\alpha^* = \beta^* = \frac{-2a^2 - 7ab - 2b^2 \mp \sqrt{a^4 + 31a^3b + 60a^2b^2 + 31ab^3 + b^4}}{3}$$

for $\lambda \neq 0$.

Therefore, $M(\alpha^*, \beta^*)$ has minimum and maximum, respectively given by

$$I_2 = \frac{-4a^2 - 14ab - 4b^2 - 2\sqrt{a^4 + 31a^3b + 60a^2b^2 + 31ab^3 + b^4}}{3} \tag{14}$$

and

$$I_3 = \frac{-4a^2 - 14ab - 4b^2 + 2\sqrt{a^4 + 31a^3b + 60a^2b^2 + 31ab^3 + b^4}}{3}.$$

In particular, since we are assuming $\alpha < \beta$ for α and β real roots of $p_2(x)$, it follows that $2\beta > M(\alpha, \beta) \geq I_2$ and therefore $\beta > J_0 > \alpha$, with $J_0 = I_2/2$. Furthermore, like $M(\alpha, \beta) \leq I_3$, it follows from the definitions of infimum and supremum that $\beta < K_0$ where $K_0 = I_3/2$. Therefore, we conclude that

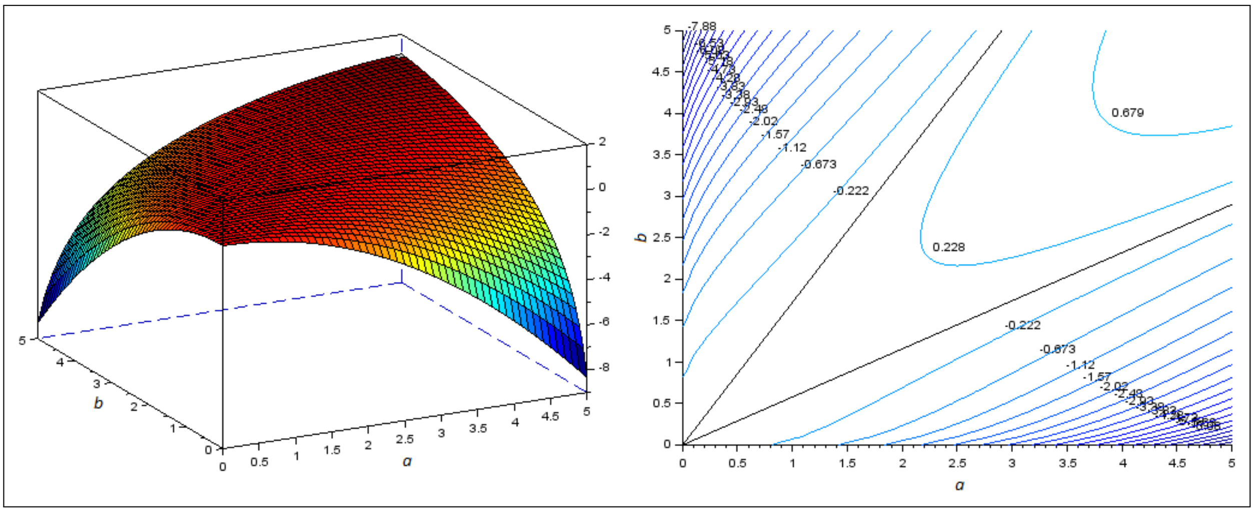
$$I_0 < \alpha < J_0 < \beta < K_0 < \gamma. \quad (15)$$

Note that according to (8) and (14) we have $I_0, J_0 < 0$, which implies $\alpha < 0$. Analyzing K_0, Figure 1, we have that $K_0 > 0$ for

$$\frac{1}{4}\left(\sqrt{13} + 1 - \sqrt{2\sqrt{13} - 2}\right) a < b < \frac{1}{4}\left(\sqrt{13} + 1 + \sqrt{2\sqrt{13} - 2}\right) a,$$

which guarantees at least $\gamma > 0$.

Figure 1 – Signal of K_0



Source: the authors (2024)

Furthermore, it is possible to state that β is also strictly negative, Figure 4. Generally speaking, α, β and γ in (15) are completely determined with Cardano-Tartaglia method by

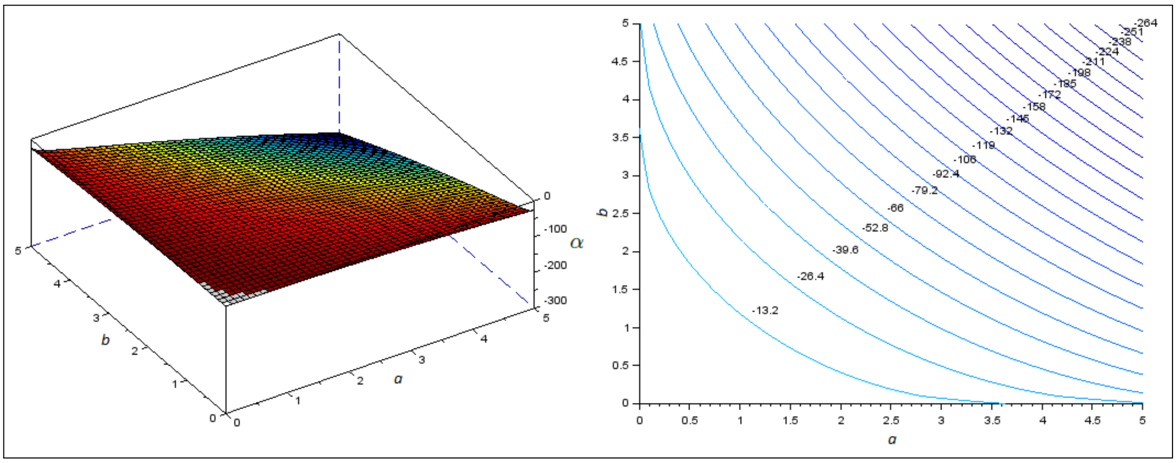
$$\begin{aligned}\alpha &= \frac{2}{3}(a+b)\sqrt{a^2+29ab+b^2}\cos\left(\frac{1}{3}\arccos\left[\frac{8a^4-508a^3b-1761a^2b^2-508ab^3+8b^4}{8(a+b)(a^2+29ab+b^2)^{3/2}}\right]+\frac{2\pi}{3}\right)\\ &- \frac{1}{3}(2a^2+7ab+2b^2),\end{aligned}$$

$$\begin{aligned}\beta &= \frac{2}{3}(a+b)\sqrt{a^2+29ab+b^2}\cos\left(\frac{1}{3}\arccos\left[\frac{8a^4-508a^3b-1761a^2b^2-508ab^3+8b^4}{8(a+b)(a^2+29ab+b^2)^{3/2}}\right]+\frac{4\pi}{3}\right)\\ &- \frac{1}{3}(2a^2+7ab+2b^2)\end{aligned}$$

and

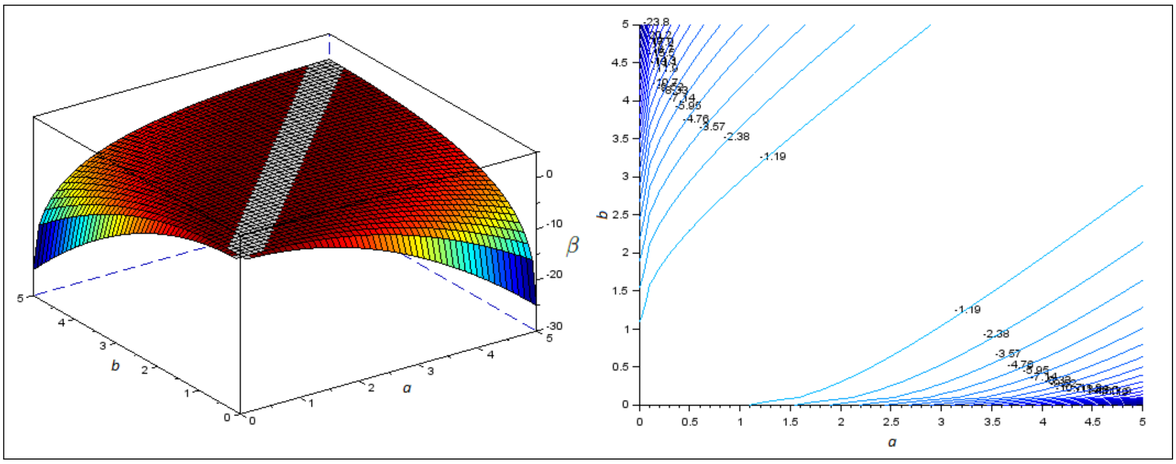
$$\begin{aligned}\gamma &= \frac{2}{3}(a+b)\sqrt{a^2+29ab+b^2}\cos\left(\frac{1}{3}\arccos\left[\frac{8a^4-508a^3b-1761a^2b^2-508ab^3+8b^4}{8(a+b)(a^2+29ab+b^2)^{3/2}}\right]\right)\\ &- \frac{1}{3}(2a^2+7ab+2b^2).\end{aligned}$$

Figure 2 – Roots α, β and γ of polynomial $p_2(x)$



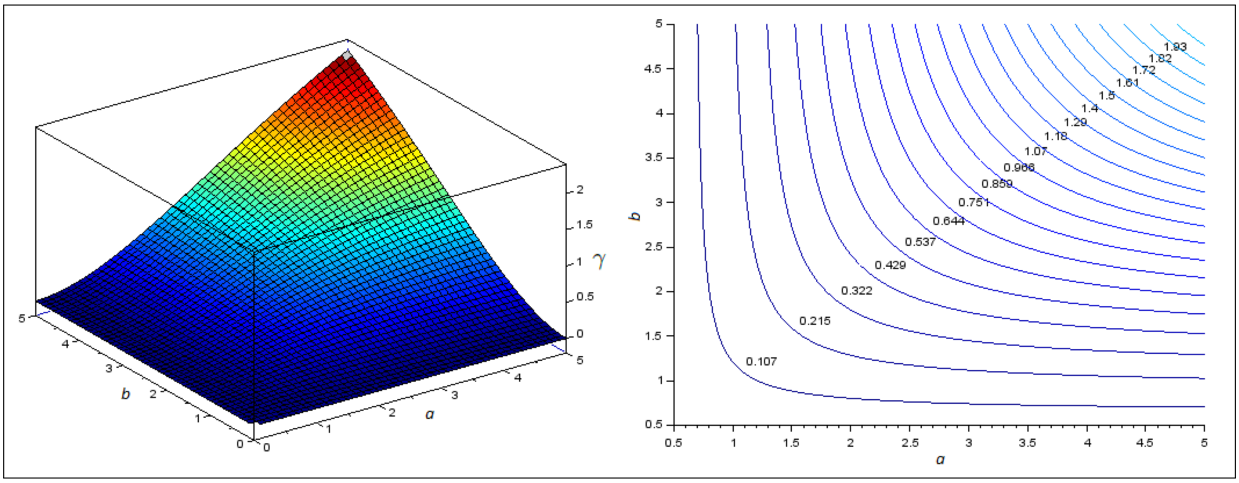
Source: the authors (2024)

Figure 3 – Roots α, β and γ of polynomial $p_2(x)$



Source: the authors (2024)

Figure 4 – Roots α, β and γ of polynomial $p_2(x)$



Source: the authors (2024)

Returning to the equation (7) we see that from the analysis established above, the polynomial $p_1(r)$ has two real roots given by $\pm\sqrt{\gamma}$. Consequently, by defining $r_0 = \sqrt{\gamma}$ according to the initial geometry of the problem, we obtain from the analysis of the discriminant Δ_1 that the equation (6) has exactly two real and distinct solutions in the case where $r = r_0$, three real and distinct solutions when $0 < r < r_0$ and a single real solution for $r > r_0$.

Thus, working with the coefficients $d_0,\ d_1$ and d_2 of the polynomial equation in (6), it is possible to obtain the following explicit solutions

$$c_0 = \frac{1}{3}\sqrt[3]{4(a+b)(2a^2-5ab+2b^2+18r^2)} + \frac{a+b}{3}$$

and

$$c_1 = -\frac{2}{3}\sqrt[3]{4(a+b)(2a^2-5ab+2b^2+18r^2)} + \frac{a+b}{3},$$

on the case $r = r_0$. For $0 < r < r_0$, we have

$$c_2 = \frac{4}{3}\sqrt{a^2-ab+b^2-3r^2}\cos\left(\frac{1}{3}\arccos\left[-\frac{(a+b)(2a^2-5ab+2b^2+18r^2)}{2(a^2-ab+b^2-3r^2)^{3/2}}\right]\right) + \frac{a+b}{3},$$

$$c_3 = \frac{4}{3}\sqrt{a^2-ab+b^2-3r^2}\cos\left(\frac{1}{3}\arccos\left[-\frac{(a+b)(2a^2-5ab+2b^2+18r^2)}{2(a^2-ab+b^2-3r^2)^{3/2}}\right] + \frac{2\pi}{3}\right) + \frac{a+b}{3}$$

and

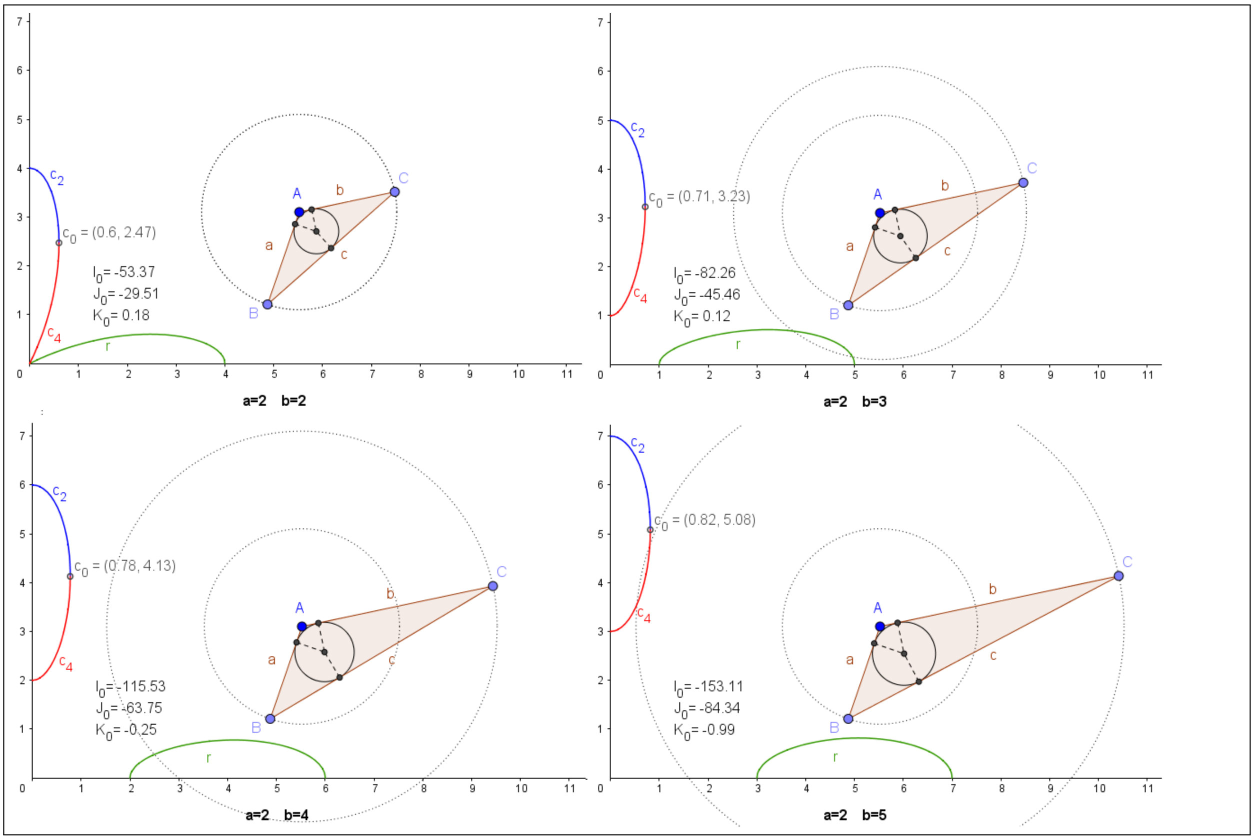
$$c_4 = \frac{4}{3}\sqrt{a^2 - ab + b^2 - 3r^2}\cos\left(\frac{1}{3}\arccos\left[-\frac{(a+b)(2a^2 - 5ab + 2b^2 + 18r^2)}{2(a^2 - ab + b^2 - 3r^2)^{3/2}}\right] + \frac{4\pi}{3}\right) + \frac{a+b}{3}.$$

On the other hand, for $r > r_0$,

$$\begin{aligned} c_5 \;=\;& \frac{1}{3}\sqrt[3]{-4(a+b)(2a^2 - 5ab + 2b^2 + 18r^2) + 24\sqrt{3p_1(r)}} \\ -\;& \frac{1}{3}\sqrt[3]{4(a+b)(2a^2 - 5ab + 2b^2 + 18r^2) + 24\sqrt{3p_1(r)}} + \frac{a+b}{3}. \end{aligned}$$

Since geometrically c denotes a side of the circumscribed triangle, we will consider in (6) only c_0, c_2 and c_4 given above, because we obtain computationally that the other explicit solutions are negative.

Figure 5 – Solutions $c_0(r)$, $c_2(r)$, $c_4(r)$ and $r(c)$ for specific values a and b of equations (6) and (4)



Source: the authors (2024)

Figure 5 presents a numerical simulation of these solutions considering four combinations of values for the sides a and b of the triangle. In addition, presents $r(c)$ given in (4) and the extremes of the existence intervals obtained in (15).

Finally, returning to the equation (1) and replacing these expressions, we obtain respectively

$$A_0 = \frac{r_0}{6}\left[\sqrt[3]{4(a+b)(2a^2-5ab+2b^2+18r_0^2)}+4(a+b)\right], \quad (16)$$

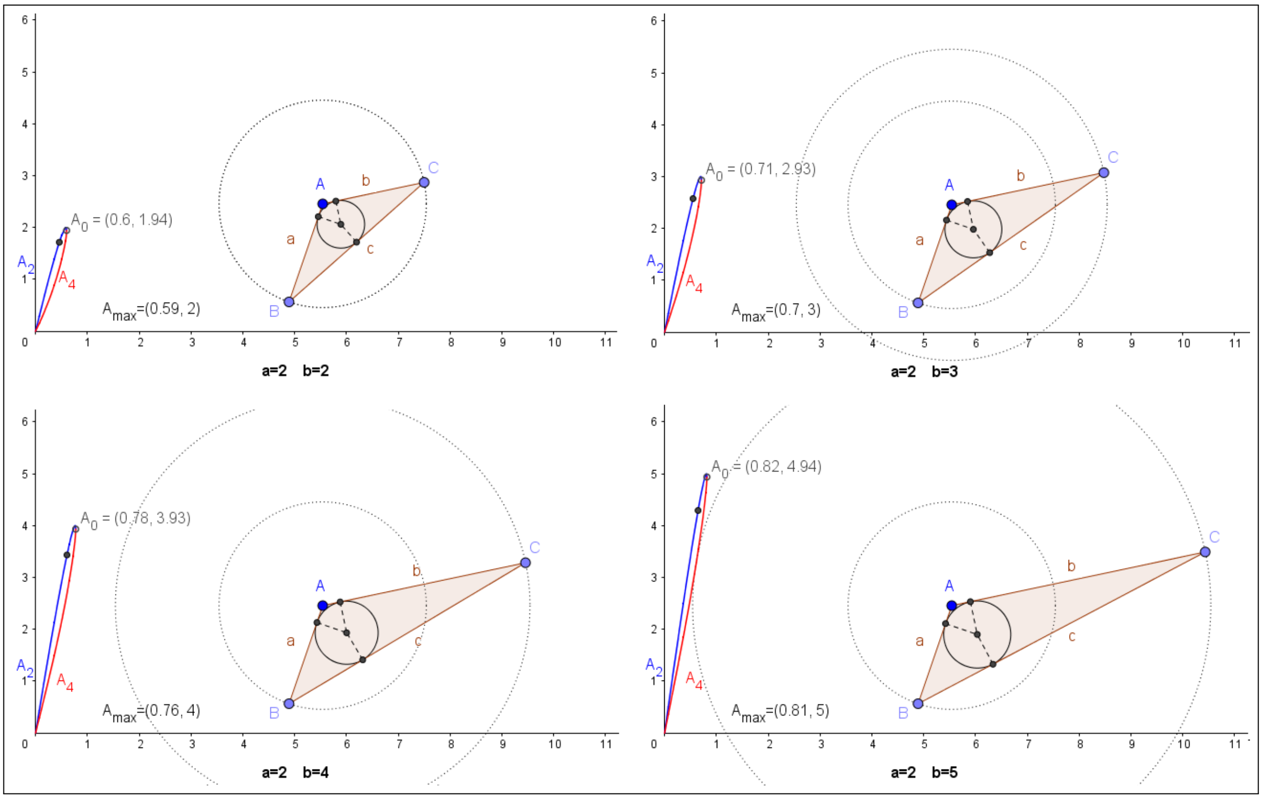
$$A_2 = \frac{2r}{3}\left\{a+b+\sqrt{a^2-ab+b^2-3r^2}\cos\left(\frac{1}{3}\arccos\left[-\frac{(a+b)(2a^2-5ab+2b^2+18r^2)}{2(a^2-ab+b^2-3r^2)^{3/2}}\right]\right)\right\} \quad (17)$$

and

$$A_4 = \frac{2r}{3}\left\{a+b+\sqrt{a^2-ab+b^2-3r^2}\cos\left(\frac{1}{3}\arccos\left[-\frac{(a+b)(2a^2-5ab+2b^2+18r^2)}{2(a^2-ab+b^2-3r^2)^{3/2}}\right]\right)+\frac{4\pi}{3}\right\} \quad (18)$$

which are the solutions that determine the triangle area, from the inscribed circle radius and from the fixed sides a and b of the triangle, Figure 6.

Figure 6 – Triangle areas depending on the inscribed circle radius



Source: the authors (2024)

Note that the largest radius of the inscribed circle in the triangle with fixed sides a e b is given by r_0, which is associated with A_0. In addition, the maximum area (A_{max}) occurs for the right triangle with hypotenuse in c and radius in (5) given by

$$r = \frac{a+b-\sqrt{a^2+b^2}}{2}. \tag{19}$$

3 RELATION BETWEEN THE AREAS OF THE TRIANGLE AND ITS INSCRIBED CIRCLE

In this section, we bring solutions that relate the triangle area (A) with the inscribed circle area $(\hat{A})$ and its inverses. The conditions of the triangle were maintained, with a and b fixed, and the methods and results obtained in the previous section were used.

Considering

$$r = \sqrt{\frac{\hat{A}}{\pi}} \tag{20}$$

where $\hat{A}$ denotes the inscribed circle area, we can write equations (16), (17) and (18) as

$$A_0 = \frac{1}{6}\sqrt{\frac{\hat{A}}{\pi}}\left[\sqrt[3]{4(a+b)(2a^2-5ab+2b^2+18\frac{\hat{A}}{\pi})}+4(a+b)\right],$$

$$\begin{aligned} A_2 &= \frac{2}{3}\sqrt{\frac{\hat{A}}{\pi}}\left\{\sqrt{a^2-ab+b^2-3\frac{\hat{A}}{\pi}}\cos\left(\frac{1}{3}\arccos\left[-\frac{(a+b)(2a^2-5ab+2b^2+18\frac{\hat{A}}{\pi})}{2(a^2-ab+b^2-3\frac{\hat{A}}{\pi})^{3/2}}\right]\right)\right. \\ &+ \left. a+b\right\} \end{aligned}$$

and

$$\begin{aligned} A_4 &= \frac{2}{3}\sqrt{\frac{\hat{A}}{\pi}}\left\{\sqrt{a^2-ab+b^2-3\frac{\hat{A}}{\pi}}\cos\left(\frac{1}{3}\arccos\left[-\frac{(a+b)(2a^2-5ab+2b^2+18\frac{\hat{A}}{\pi})}{2(a^2-ab+b^2-3\frac{\hat{A}}{\pi})^{3/2}}\right]+\frac{4\pi}{3}\right)\right. \\ &+ \left. a+b\right\}. \end{aligned}$$

Substituting equation (20) in equation (19), we have that the maximum area of triangle occurs when

$$\hat{A} = \frac{\pi}{4}\left(a + b - \sqrt{a^2 + b^2}\right)^2. \tag{21}$$

From the equation $\hat{A} = \pi r^2$ with $r = r_0$, we obtain the area of the largest inscribed circle in the triangle with fixed sides a and b, as being

$$\begin{aligned} \hat{A}_0 &= \frac{\pi}{3}\left\{2(a+b)\sqrt{a^2+29ab+b^2}\cos\left(\frac{1}{3}\arccos\left[\frac{8a^4-508a^3b-1761a^2b^2-508ab^3+8b^4}{8(a+b)(a^2+29ab+b^2)^{3/2}}\right]\right)\right. \\ &- \left.(2a^2+7ab+2b^2)\right\}. \end{aligned}$$

In the search for the relation of inverse dependence, that is, the circle area $(\hat{A})$ depending on the triangle area (A), we write equation (2) as

$$16A^2 = (a+b+c)(b+c-a)(a+c-b)(a+b-c),$$

or else as a quartic equation in c,

$$c^4 - 2(a^2+b^2)c^2 + (a^2-b^2)^2 + 16A^2 = 0.$$

This last equation can be solved by changing the variable $x = c^2$. According to the geometry of the problem, as c denotes a side of triangle, we highlight as solutions

$$c_6 = \sqrt{a^2+b^2+2\sqrt{a^2b^2-4A^2}}$$

and

$$c_7 = \sqrt{a^2+b^2-2\sqrt{a^2b^2-4A^2}}.$$

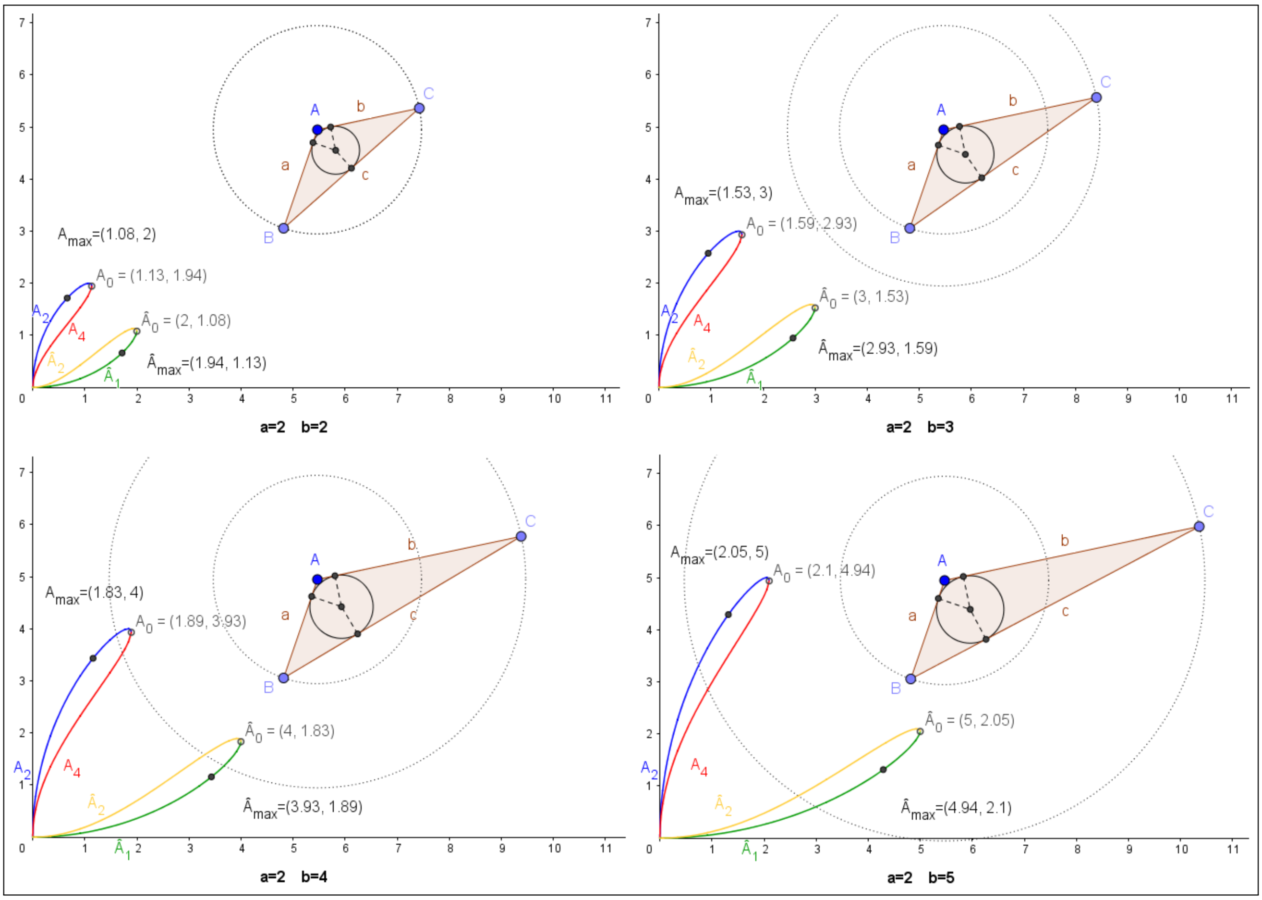
Finally, working with the equations in (1) and (20), together with c_6 and c_7 obtained above, we have respectively the relations between the areas of the inscribed circle and the triangle, Figure 7, by

$$\hat{A}_1 = \frac{4\pi A^2}{\left(a+b+\sqrt{a^2+b^2+2\sqrt{a^2b^2-4A^2}}\right)^2}$$

and

$$\hat{A}_2 = \frac{4\pi A^2}{\left(a+b+\sqrt{a^2+b^2-2\sqrt{a^2b^2-4A^2}}\right)^2}.$$

Figure 7 – Triangle area $A(\hat{A})$ and circle area $\hat{A}(A)$ for specific values a and b



Source: the authors (2024)

4 CONCLUSION

In the search for relations between the triangle area with two fixed sides and the radius of the inscribed circle, we found cosine solutions, which arose from the solution of a cubic polynomial equation relating two well-known equations of the triangle area.

Analyzing the discriminant of the cubic equation, we were able to establish intervals of existence for its roots and ensure that one of them is positive. With that, it was possible to obtain all the solutions for the cubic equation, which allowed to present the relations $A(r)$ between the triangle area with two fixed sides and the radius of the inscribed circle and also the relations $A(\hat{A})$ between the areas of the triangle and the inscribed circle. Furthermore, in the search for the inverse dependence $\hat{A}(A)$, a quartic polynomial equation was solved by a simple variable substitution.

Finally, we highlighted the presentation of numerical simulations of specific cases for sides a and b of the triangle and also points that represent both the largest area of the triangle and the largest inscribed circle.

For future studies, we are already working to understand the relations between the triangle and other circles, such as the circumscribed circle and escribed circle.

REFERENCES

Brooks, R. & Waksman, P. (1987). The first eigenvalue of a scalene triangle. *Proceedings of the American Mathematical Society*, 1(100):175–182. doi: https://doi.org/10.2307/2046142.

Buck, M. W. & Siddon, R. L. (2012). The area of a polygon with an inscribed circle. *Metric Geometry*. doi: https://doi.org/10.48550/arXiv.1203.3438.

Cardano, G. (1993). *Ars Magna or The Rules of Algebra*. New York: Dover Publications. Translated and edited by Richard Witmer.

Evans, A. B. (1874). The plane triangle and its six circles. *The Analyst*, pages 189–194. doi: https://doi.org/10.2307/2636203.

Girard, A. (1884). *Invention nouvelle en l'algèbre*. Bierens de Haan (ed.) Leiden.

Hansen, D. W. (2003). On inscribed and escribed circles of right triangles, circumscribed triangles, and the four-square, three-square problem. *The Mathematics Teacher*, (96):358–364.

Lima, E. L. (1987). A equação do terceiro grau. *Matemática Universitária*, (5):9–23.

Maley, F. M., Robbins, D. P., & Roskies, J. (2005). On the areas of cyclic and semicyclic polygons. *Advances in Applied Mathematics*, (34):669–689. doi: https://doi.org/10.1016/j.aam.2004.09.008.

Nunes, K. (2013). Inscribed circles and spheres. *The Mathematical Gazette*, (97):290–294.

Oliver, B. M. (1993). Heron's remarkable triangle area formula. *The Mathematics Teacher*, (86):161–163.

Robbins, D. P. (1994). Areas of polygons inscribed in a circle. *Discrete and Computational Geometry*, (12):223–236. doi: https://doi.org/10.1007/BF02574377.

Tartaglia, N. (1554). *Quesiti et Inventioni Diverse*. Brescia.

Viète, F. (1646). *Opera mathematica*. F. van Schouten (ed.). Leiden.

Author contributions

1 – Radael de Souza Parolin (Corresponding Author)
Mathematician, Professor
https://orcid.org/0000-0003-3803-8920 • radaelparolin@unipampa.edu.br
Contribution: Conceptualization; Methodology; Literature Review; Data Analysis; Writing – Original Draft Preparation, Review & Editing.

2 – Alisson Darós Santos
Mathematician, Professor
https://orcid.org/0000-0002-2082-6999 • alissondaros@unipampa.edu.br
Contribution: Conceptualization; Methodology; Literature Review; Data Analysis; Writing – Original Draft Preparation, Review & Editing.

How to cite this article

Parolin, R. de S., & Santos, A. D. (2024). Trigonometric solutions that relate geometrical quantities of the triangle and the inscribed circle. *Ciência e Natura*, Santa Maria, v.46, e75968. DOI: https://doi.org/10.5902/2179460X75968.