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Special Edition

A preliminary study on the application of the two-space nonperiodic asymptotic homogenization method to the EEG forward problem with continuously differentiable coefficient

Um estudo preliminar sobre a aplicação do método de homogeneização assintótica não periódica de dois espaços ao problema direto do EEG com coeficiente continuamente diferenciável

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ABSTRACT

The knowledge on the Electroencephalogram (EEG) forward problem is important to improve the imaging of the neural activity, that is an inverse problem. This paper introduces the study of the EEG forward problem via a nonperiodic homogenization technique: the two-space method. Considering a concentric spheres head model for the brain-skull-scalp medium as an micro- heterogeneous medium, a simplification consisting of a 1D problem in spherical coordinates with continuously differentiable coefficient is considered. The two-space method is applied successfully, and a numerical example shows the convergence of the micro-heterogeneous solution to the one obtained by the two-space method, as was expected. The preliminary conclusion is that this approach for the EEG forward problem with homogenization techniques shows itself as very promising. More experiments should be executed, considering more realistic models for the head.

Keywords: Non-periodic Asymptotic Homogenization, Two-Space Method, Micro-heterogeneous Media, Electroencephalogram, Inverse Problems

RESUMO

Entender o problema direto para o Eletroencefalograma (EEG) é importante para a obtenção de imagens da atividade neural, processo que constitui um problema inverso. Este trabalho introduz o estudo do problema direto para o EEG utilizando técnicas de homogeneização não-periódica, o método de dois espaços. Representando a cabeça (um meio cérebro-crânio-escalpo) através de esferas concêntricas, como um meio micro-heterogêneo, simplifica-se o problema para um unidimensional em coordenadas esféricas, e coeficientes continuamente diferenciáveis. O método é aplicado com sucesso, e por meio de um exemplo numérico mostra-se a convergência da solução micro-heterogênea para a solução obtida através da homogeneização com o método de dois espaços, como era esperado. Conclui-se que essa abordagem para o problema direto do EEG através de métodos de homogeneização se mostrou promissora. Mais experimentos devem ser executados, considerando modelos mais realísticos para a cabeça.

Palavras-chave: Homogeneização Assintótica Não-Periódica, Método dos dois espaços, Meios Micro-heterogêneos, Eletroence- falograma, Problemas Inversos.

1 INTRODUCTION

Electroencephalography (EEG) is an imaging technique that allows the study of the neural dynamics, with a good temporal resolution, from measurements of the electric potentials over the scalp. The imaging of the neural activity is termed inverse problem in EEG (Hallez *et al.,* 2007) and has attracted increasing interest in the neuroscience community, since it has enabled substantial advances in our understanding of the human brain function.

Due to the ill-posedness of the inverse problem in EEG (Morales *et al.*, 2019), a successfully EEG imaging procedure (determination of the neural activity) requires a reduction of the noise signal acquisition (the electrical potentials measured by electrodes placed over the scalp associated to the bipolar current inputs) and a detailed information about the geometry and physical properties of the head tissues that are between the neural activity sources and the EEG sensors. In other words, it requires an accurate (and computationally low-cost) solution to the so-called EEG forward problem.

Consider a domain $\Omega \subset \mathbb{R}^3$ with boundary Γ_Ω, occupied by an heterogeneous anisotropic conductive medium with local conductivity $\sigma(x,y,z)$ representing the head, so (*x,y,z*) $\in \Omega$ is a voxel position within the head. With such considerations, the EEG forward problem studied here is given by the quasi-static Maxwell equation subject to conditions for no outgoing normal electric fluxes $(\sigma \cdot \nabla\phi)\cdot \hat{e}_n$ on the surface of the head (scalp) Γ_Ω and for the electric potential ϕ on the electrodes (Morales et al.,2019), governed by

$$\begin{cases} \nabla \cdot (\sigma(x,y,z) \cdot \nabla\phi(x,y,z)) = -\nabla \cdot J(x_{if},y_{if},z_{if}), \quad (x,y,z) \in \Omega \\ [(\sigma \cdot \nabla\phi) \cdot \hat{e}_n]\,|_{\Gamma_\Omega} = 0, \qquad \phi|_{\Gamma_\Omega} = \phi_\Gamma \end{cases} \tag{1}$$

where $J(x_{if},y_{if},z_{if})$ is the electric current density in positions $(x_{if},y_{if},z_{if}) \in \Omega$.

Since, in general, no analytical solution ϕ *(X, Y, Z)* of problem (1) is available, various computational approaches have been proposed. Boundary element methods (BEMs) are among the most popular due to their low computational requirements, but they suffer from limitations on the modeling as they are able to consider only piecewise isotropic head subdomains (brain, skull and scalp). On the other hand, the anisotropy is considered in the numerical approaches using finite element methods (FEMs), finite difference methods (FDMs) and finite volume methods (FVMs). Those approaches require that the sizes of the mesh elements (generally tetrahedra) must be chosen as small as the smallest anisotropy variation, resulting in a large number of unknowns in sparse linear equation systems and, consequently, in a extremely high computational cost, e.g., Piastra *et al.* (2018); Engwer *et al.* (2017) and references therein.

In this context, this work brings the application of mathematical homogenization methods. Owing to the micro-heterogeneity of the brain and assuming that it is locally continuous, the equivalent homogeneity hypothesis allows considering these methods for producing a hypothetical homogeneous material which is physically equivalent to the micro-heterogeneous one, and is modeled with differential equations with constant coefficients instead of the original model with rapidly oscillating coefficients that represent the micro-heterogeneous properties of the brain. The main idea is that the solution of the problem for the hypothetical homogeneous material satisfactorily approximates the solution for the micro-heterogeneous material, with the rapidly oscillating coefficient. See Nandakumaran (2007) for an overview of the homogenization techniques.

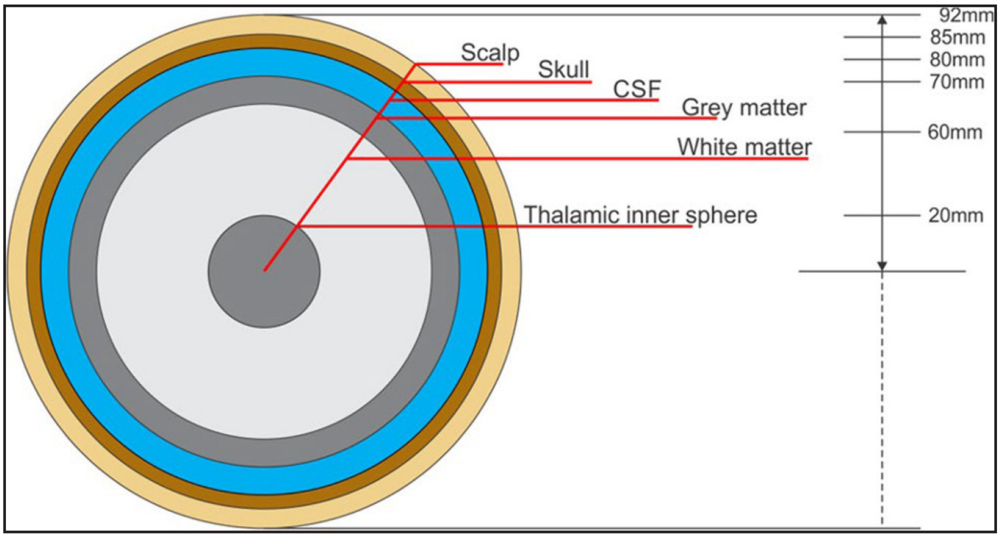
Great advances have been achieved for the case of microperiodic media via the asymptotic homogenization method – AHM (Bakhvalov and Panasenko, 1989; Panasenko, 2008). Some results are in Décio Jr *et al.* (2019). An alternative for nonperiodic media is a generalization of the AHM: the so-called two-space method – TSM, which was originally presented and applied to the Navier-Stokes equation by Keller (1980). Recently, Suárez *et al.* (2016) studied the application of the TSM to a family of problems with elliptic equations considering piecewise continuously differentiable coefficients which model the physical behavior of composite materials.

In this contribution, we investigate the effect of the two-space homogenization on the numerical solution of an 1D forward EEG problem. The adopted homogenization strategy splits the forward anisotropic EEG problem into tree isotropic equations for which either an analytical solution exists or the numerical effort is comparable to that of BEM. Some considerations on the methodology of the TSM to this EEG problem are presented. Due to the fact that, to the best of our knowledge, there is just a few results on the TSM in the current literature, this work finds its originality in the application of the TSM in the EEG context.

2 THE RADIAL HEAD MODEL

The head model considered here is based on the concentric spheres model (accounting for the different tissues in the head as different layers – see, Figure 1) and assuming radially symmetrical conductivity $\sigma(x,y,z) = \sigma(\rho)$, $\rho = \sqrt{x^2 + y^2 + z^2}$. In fact, the first head model used was an homogeneous sphere, evolving to multilayered spheres until the emergence of finite elements models (Hallez et al., 2007). In this initial work on the application of the TSM to the EEG forward problem, we consider that the conductivity $\sigma^\varepsilon(\rho) = \sigma(\rho/\varepsilon)$ is a continuously differentiable, positive and bounded function which oscillates rapidly in terms ofthe geometric parameter ε, $0 < \varepsilon \ll 1$, accounting for the separation of structural scales and characterizing the micro-heterogeneityof the head.

Figure 1 – The concentric spheres head's model



Source:Morales et al. (2019)

Under such considerations, the EEG forward problem (1) can be rewritten in spherical coordinates as

$$\begin{cases} \dfrac{\partial}{\partial \rho}\left(\sigma^{\varepsilon}(\rho)\dfrac{\partial \phi^{\varepsilon}}{\partial \rho}\right) + \dfrac{2}{\rho}\sigma^{\varepsilon}(\rho)\dfrac{\partial \phi^{\varepsilon}}{\partial \rho} = f^{\varepsilon}(\rho), \quad \rho \in (0,l) \\ \dfrac{\partial \phi^{\varepsilon}}{\partial \rho}|_{\rho=l} = 0, \qquad \phi^{\varepsilon}|_{\rho=l} = \phi_l \end{cases}, \tag{2}$$

where $f^{\varepsilon}(\rho) = f(\rho, \rho/\varepsilon)$. Applying the Green's function theory (Folland, 1992), it is possible to show that the unique solution for $\phi^{\varepsilon}(\rho) \in C^2([0,l])$ problem (2) is given by

$$\phi^{\varepsilon}(\rho) = \int_{\rho}^{l} \frac{1}{\eta^2 \sigma^{\varepsilon}(\eta)} \left(\int_{\eta}^{l} \tau^2 f^{\varepsilon}(\tau) d\tau \right) d\eta + \phi_l$$

3 THE TWO-SPACE METHOD APPLIED TO THE FORWARD EEG PROBLEM

The two-space homogenization method (Keller, 1980) seeks for a formal asymptotic solution (FAS) of problem (2) which, because of the linearity of the differential equation and the availability of an estimate resulting from a maximum principle (Bakhvalov and Panasenko, 1989; Larsson and Thomée, 2009), is an asymptotic expansion of its exact solution $\phi^{\varepsilon}(\rho)$. Such a FAS is a two-scale series in terms of powers of ε. In particular, consider the following FAS:

$$\phi^{\varepsilon}(\rho) \sim \phi^{(2)}(\rho,\varepsilon) = \phi_0(\rho) + \varepsilon\phi_1(\rho,\gamma) + \varepsilon^2\phi_2(\rho,\gamma), \quad \gamma = \frac{\rho}{\varepsilon}, \tag{3}$$

where the local and global variables γ and ρ represent the microscale and the macroscale, respectively, and the unknown functions $\phi_0(\rho) \in C^2([0,l])$, and $\phi_1(\rho,\gamma), \phi_2(\rho,\gamma) \in C^2([0,l] \times [0,l/\varepsilon])$ are bounded

Substitute (3) into the equation of problem (2) and observe that in order to $\phi^{(2)}(\rho,\varepsilon)$ be a FAS of the problem, that is,

$$\frac{\partial}{\partial\rho}\left(\sigma(\gamma)\frac{\partial\phi^{(2)}}{\partial\rho}\right) + \frac{2}{\rho}\sigma(\gamma)\frac{\partial\phi^{(2)}}{\partial\rho} - f(\rho,\gamma) = O(\varepsilon^2),$$

the following recurrence of equations must be satisfied:

$$\frac{\partial}{\partial\gamma}\left[\sigma(\gamma)\left(\frac{d\phi_0}{d\rho} + \frac{\partial\phi_1}{\partial\gamma}\right)\right] = 0, \tag{4}$$

$$\frac{\partial}{\partial\gamma}\left[\sigma(\gamma)\left(\frac{\partial\phi_1}{\partial\rho} + \frac{\partial\phi_2}{\partial\gamma}\right)\right] = -\sigma(\gamma)\left[\frac{d^2\phi_0}{d\rho^2} + \frac{\partial^2\phi_1}{\partial\rho\partial\gamma} + \frac{2}{\rho}\left(\frac{d\phi_0}{d\rho} + \frac{\partial\phi_1}{\partial\gamma}\right)\right] + f(\rho,\gamma) \tag{5}$$

where the chain rule $\frac{\partial}{\partial\rho} = \frac{\partial}{\partial\rho} + \frac{1}{\varepsilon}\frac{\partial}{\partial\gamma}$ was used.

Integrate (4) twice with respect to *y* in the interval *[y0, y]* ⊂ *(0, ∞)*, to find $\phi_1(\rho, \gamma)$ as

$$\phi_1(\rho,\gamma) = \phi_1(\rho,\gamma_0) + \sigma(\gamma_0)\left(\frac{d\phi_0}{d\rho} + \left.\frac{\partial\phi_1}{\partial\gamma}\right|_{\gamma=\gamma_0}\right)\int_{\gamma_0}^{\gamma}\frac{d\tau}{\sigma(\tau)} - \frac{d\phi_0}{d\rho}(\gamma-\gamma_0)$$

which turns into

$$\frac{\phi_1(\rho,\gamma) - \phi_1(\rho,\gamma_0)}{\gamma-\gamma_0} = \sigma(\gamma_0)\left(\frac{d\phi_0}{d\rho} + \left.\frac{\partial\phi_1}{\partial\gamma}\right|_{\gamma=\gamma_0}\right)\frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma}\frac{d\tau}{\sigma(\tau)} - \frac{d\phi_0}{d\rho} \tag{6}$$

Assume that the limit in (7) exists and is independent of γ_0 so the so-called $\hat{\sigma}$ effective coefficient $\hat{\sigma}$ can be defined

$$\hat{\sigma} = \left[\lim_{\gamma\to\infty}\left(\frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma}\frac{d\tau}{\sigma(\tau)}\right)\right]^{-1} \tag{8}$$

Isolate $\left.\frac{\partial\phi_1}{\partial\gamma}\right|_{\gamma=\gamma_0}$ in (7) considering $\hat{\sigma}$ form (8) to obtain

$$\left.\frac{\partial\phi_1}{\partial\gamma}\right|_{\gamma=\gamma_0} = \left(\frac{\hat{\sigma}}{\sigma(\gamma_0)} - 1\right)\frac{d\phi_0}{d\rho} \tag{9}$$

Observe that $\gamma_0 \in (0,\infty)$ is arbitrary, so integration of (9) with respect to $\gamma \equiv \gamma_0$ yields

$$\phi_1(\rho,\gamma) = \frac{d\phi_0}{d\rho}\int_0^{\gamma}\left(\frac{\hat{\sigma}}{\sigma(\tau)} - 1\right)d\tau + \phi_1(\rho,0) \tag{10}$$

Take such considerations into account, 5 turns into

$$\frac{\partial}{\partial\gamma}\left[\sigma(\gamma)\frac{\partial\phi_2}{\partial\gamma}\right] = -\frac{d^2\phi_0}{d\rho^2}\frac{\partial}{\partial\gamma}\left[\sigma(\gamma)N_1(\gamma)\right] - \hat{\sigma}\left[\frac{d^2\phi_0}{d\rho^2} + \frac{2}{\rho}\frac{d\phi_0}{d\rho}\right] + f(\rho,\gamma) \tag{11}$$

where the so-called local solution $N_1(\gamma)$ is defined as where the so-called local solution $N_1(\gamma)$ is defined as

$$N_1(\gamma) = \int_0^\gamma \left(\frac{\hat{\sigma}}{\sigma(\tau)} - 1 \right) d\tau$$

Integrate (4) twice with respect to γ in the interval $[\gamma_0, \gamma] \subset (0, \infty)$ to obtain

$$\sigma(\gamma)\frac{\partial \phi_2}{\partial \gamma} - \sigma(\gamma_0)\left.\frac{\partial \phi_2}{\partial \gamma}\right|_{\gamma=\gamma_0} = -\frac{d^2\phi_0}{d\rho^2}\left[\sigma(\gamma)N_1(\gamma) - \sigma(\gamma_0)N_1(\gamma_0)\right] - (\gamma-\gamma_0)\,\hat{\sigma}\left[\frac{d^2\phi_0}{d\rho^2} + \frac{2}{\rho}\frac{d\phi_0}{d\rho}\right] + \int_{\gamma_0}^{\gamma} f(\rho,\tau) \quad (12)$$

Divide (12) by $\gamma - \gamma_0$ and take the limit when $\gamma \to \infty$ recalling that $\phi_1(\rho,\gamma)$ and $\phi_2(\rho,\gamma)$ are bounded to obtain the so-called homogenized equation

$$\hat{\sigma}\left[\frac{d^2\phi_0}{d\rho^2} + \frac{2}{\rho}\frac{d\phi_0}{d\rho}\right] = \lim_{\gamma\to\infty}\frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma} f(\rho,\tau)\,d\tau \quad (13)$$

where the limit on the right-hand side is assumed to exist independent of γ_0. It follows that (11) turns into

$$\frac{\partial}{\partial \gamma}\left[\sigma(\gamma)\frac{\partial \phi_2}{\partial \gamma}\right] = -\frac{d^2\phi_0}{d\rho^2}\frac{\partial}{\partial \gamma}\left[\sigma(\gamma)N_1(\gamma)\right] + f_0(\rho,\gamma) \quad (1)$$

where

$$f_0(\rho,\gamma) = f(\rho,\gamma) - \lim_{\gamma\to\infty}\frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma} f(\rho,\tau)\,d\tau.$$

Observe that if $f(\rho,\gamma) \equiv f(\rho)$, which means that the source is independent of the microscale, then $f_0 \equiv 0$. Operate as above to obtain

$$\phi_2(\rho,\gamma) = \frac{d^2\phi_0}{d\rho^2}\int_0^\gamma \left[\frac{\hat{N}_1\,\hat{\sigma}}{\sigma(\tau)} - N_1(\tau)\right] d\tau - \int_0^\gamma \frac{\hat{f}_{0,\sigma}\,\hat{\sigma}}{\sigma(\tau)} d\tau + \phi_2(\rho,0) \quad (14)$$

where $\hat{N}_1$ and $\hat{f}_{0,\sigma}(\rho)$ are independent of γ_0 and defined as

$$\hat{N}_1 = \lim_{\gamma\to\infty} \frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma} N_1(\tau)d\tau \quad \text{and} \quad \hat{f}_{0,\sigma}(\rho) = \lim_{\gamma\to\infty} \frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma} \frac{1}{\sigma(\tau)}\left(\int_{\gamma_0}^{\tau} f_0(\rho,\psi)d\psi\right)d\tau$$

Again, observe that if $f(\rho,\tau) \equiv f(\rho)$, then $\hat{f}_{0,\sigma} \equiv 0$.

On the other hand, substitute FAS (3) into the conditions of problem (2) to obtain

$$\phi_0|_{\rho=l} = \phi_l, \quad \left.\frac{d\phi_0}{d\rho}\right|_{\rho=l} = 0. \tag{15}$$

$$\phi_1|_{\rho=l} = 0, \quad \left.\frac{\partial\phi_1}{\partial\rho}\right|_{\rho=l} = 0 \tag{16}$$

$$\phi_2|_{\rho=l} = 0, \quad \left.\frac{\partial\phi_2}{\partial\rho}\right|_{\rho=l} = 0 \tag{17}$$

The homogenized equation (13) completed with conditions (15) define the so-called homogenized problem whose solution $\phi_0(\rho)$ completes the expressions (10) and (14) of $\phi_1(\rho,\gamma)$ and $\phi_2(\rho,\gamma)$ together with conditions (16) and (17), respectively.

The solution $\phi_0(\rho)$ of the homogenized problem (13) and (15) provides a good approximation of the exact solution $\phi^\varepsilon(\rho)$ of problem (2) and it is sufficient to consider it as the FAS in many applications. However, if knowledge of the local details of $\phi^\varepsilon(\rho)$ is required, then $\phi_1(\rho,\gamma)$, and sometimes also $\phi_2(\rho,\gamma)$, must be considered in the FAS (see Décio Jr *et al.* (2019) for a related study in a periodic nonlinear setting).

The physical meaning of the homogenized problem with solution $\phi_0(\rho)$ is that it models the behavior of the hypothetical homogeneous material with property $\hat{\sigma}$ that is physically equivalent to the micro-heterogeneous material with rapidly oscillating property $\sigma^\varepsilon(\rho)$ modeled by the original problem with solution $\phi^\varepsilon(\rho)$, so $\phi_0(\rho)$ is a good approximation of $\phi^{\varepsilon}(\rho)$ for sufficiently small ε, that is, $\phi^{\varepsilon}(\rho) \to \phi_0(\rho)$ in some sense as $\varepsilon \to 0^+$. In the next section, this convergence is numerically illustrated.

4 RESULTS

4.1 On the existence of the limit $\hat{F} = \lim_{\gamma\to\infty} \frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma} F(\tau)d\tau$

Various results of the application of the TSM described in the previous section depend on the existence of the limit

$$\hat{F} = \lim_{\gamma\to\infty} \frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma} F(\tau)d\tau \tag{18}$$

Observe that the limit (18) produces an ∞/∞ indeterminate form, so its existence is stated in the following proposition:

Proposition 1 (From the authors). Let $m, M \in \mathbb{R}_+$ be constants and $F(\gamma) \in C(\mathbb{R}_+)$ such that $0 < m < F(\gamma) < M < \infty$. Then, a sufficient condition for the existence of limit (18) is the existence of the limit $\lim_{\gamma\to\infty} F(\gamma)$, and

$$\hat{F} = \lim_{\gamma\to\infty} \frac{1}{\gamma-\gamma_0}\int_{\gamma_0}^{\gamma} F(\tau)d\tau = \lim_{\gamma\to\infty} F(\gamma) \tag{19}$$

Proof. The Fundamental Theorem of Calculus (Demidovich et al., 1987) ensures that, being F continuous in $(0,\infty)$, there exists its antiderivative $\mathcal{F}(\gamma) = \int_{\gamma_0}^{\gamma} F(\tau)d\tau$, continuous in $[\gamma 0, \gamma]$ and differentiable in $(\gamma 0, \gamma)$ such that $F'(\gamma) = F(\gamma)$.

The fact that $0 < m < F(\gamma)$ in $(0,\infty)$ implies that $\int_{\gamma_0}^{\gamma} m d\tau < \int_{\gamma_0}^{\gamma} F(\tau)d\tau$, $[\gamma_0, \gamma] \subset (0,\infty)$, so

$$m(\gamma - \gamma_0) < \int_{\gamma_0}^{\gamma} F(\tau)d\tau = \mathcal{F}(\gamma), \tag{20}$$

which implies that $F(\gamma) \to \infty$ as $\gamma \to \infty$ due to the inequality in (20). Then, the ratio in the first limit in (19) is an ∞/∞ indeterminate form, so L'Hôpital's rule (Demidovich et al., 1987) can be applied to yield

$$\hat{F} = \lim_{\gamma\to\infty} \frac{1}{(\gamma-\gamma_0)'} \left(\int_{\gamma_0}^{\gamma} F(\tau)d\tau\right)' = \lim_{\gamma\to\infty} F(\gamma)$$

This result supports the next section.

4.2 Numerical example

Consider the following realization of the EEG forward problem (2):

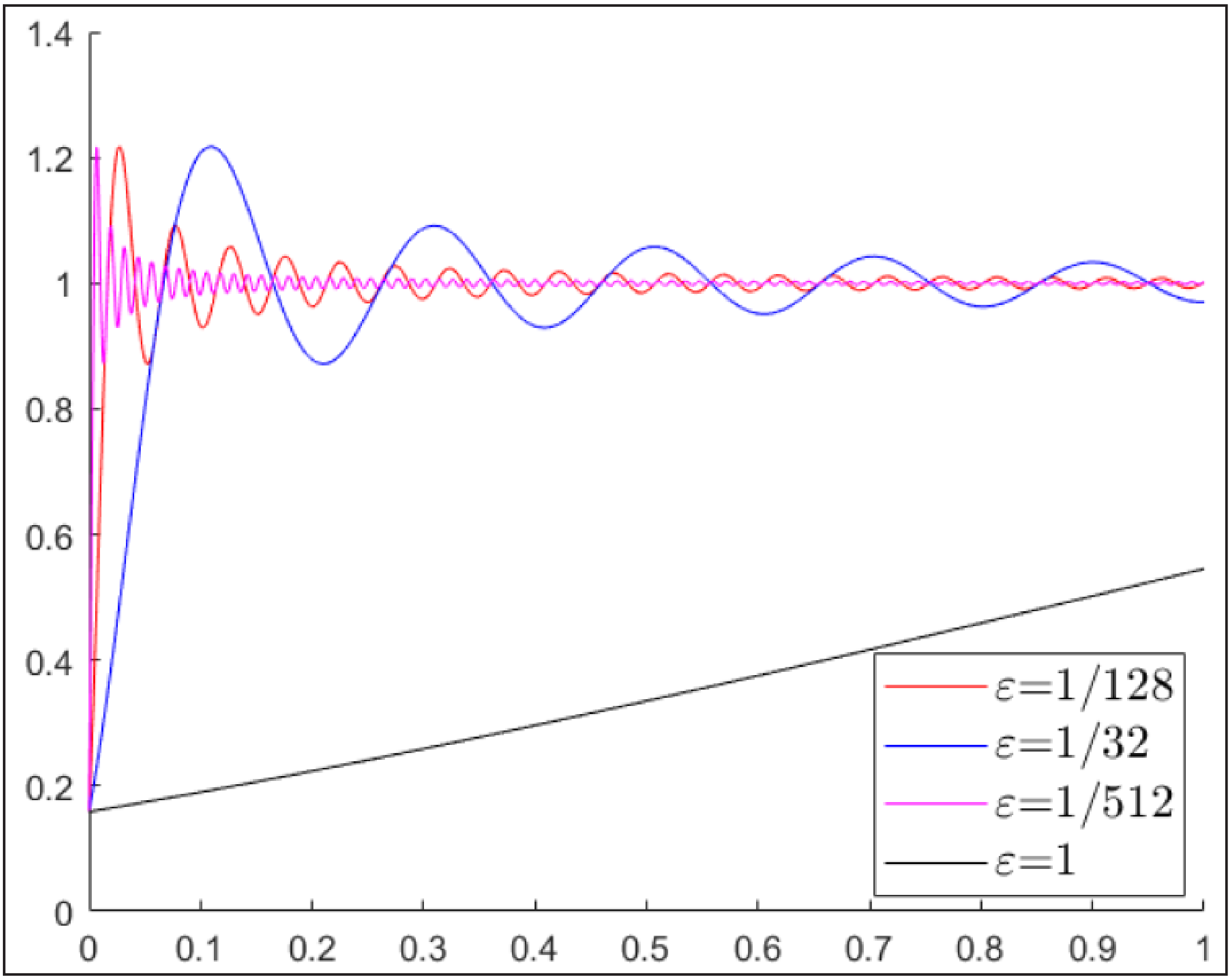
$$\begin{cases} \dfrac{\partial}{\partial\rho}\left(\sigma^{\varepsilon}(\rho)\dfrac{\partial\phi^{\varepsilon}}{\partial\rho}\right) + \dfrac{2}{\rho}\sigma^{\varepsilon}(\rho)\dfrac{\partial\phi^{\varepsilon}}{\partial\rho} = -1, \quad \rho \in (0,1), \\ \sigma^{\varepsilon}(\rho) = 1 - \dfrac{\sin\left(\dfrac{\rho}{\varepsilon}+1\right)}{\dfrac{\rho}{\varepsilon}+1}, \\ \dfrac{\partial\phi^{\varepsilon}}{\partial\rho}|_{\rho=1} = 0, \quad |\phi|_{\rho=1} = 1 \end{cases} \tag{21}$$

Observe that $\sigma^{\varepsilon}(\rho) = \sigma(\gamma)$ satisfies the conditions of Proposition 1, so does $1/\sigma(\gamma)$. Then, the effective conductivity $\hat{\sigma}$ is

$$\hat{\sigma} = \left[\lim_{\gamma\to\infty}\left(1 - \frac{\sin(\gamma+1)}{\gamma+1}\right)^{-1}\right]^{-1} = 1 \tag{22}$$

Figure 2 presents the rapidly oscillating behavior of the electrical conductivity $\sigma^{\varepsilon}(\rho)$ in (21) which increases as $\varepsilon \to 0^+$. Also, observe that such oscillations occur around $\hat{\sigma} = 1$ and their amplitudes decrease as $\varepsilon \to 0^+$, so $\sigma^{\varepsilon}(\rho)$ as $\varepsilon \to 0^+$ as expected.

Figure 2 – Behavior of $\sigma^{\varepsilon}(\rho)$ in (21)



Source: from the authors

Taking (22) into account, the homogenized problem corresponding to problem (21) is

$$\begin{cases} \dfrac{d^2\phi_0}{d\rho^2} + \dfrac{2}{\rho}\dfrac{d\phi_0}{d\rho} = -1, & \rho \in (0,1) \\ \dfrac{d\phi_0}{d\rho}|_{\rho=1} = 0, \quad \phi_0|_{\rho=1} = 1 \end{cases} \tag{23}$$

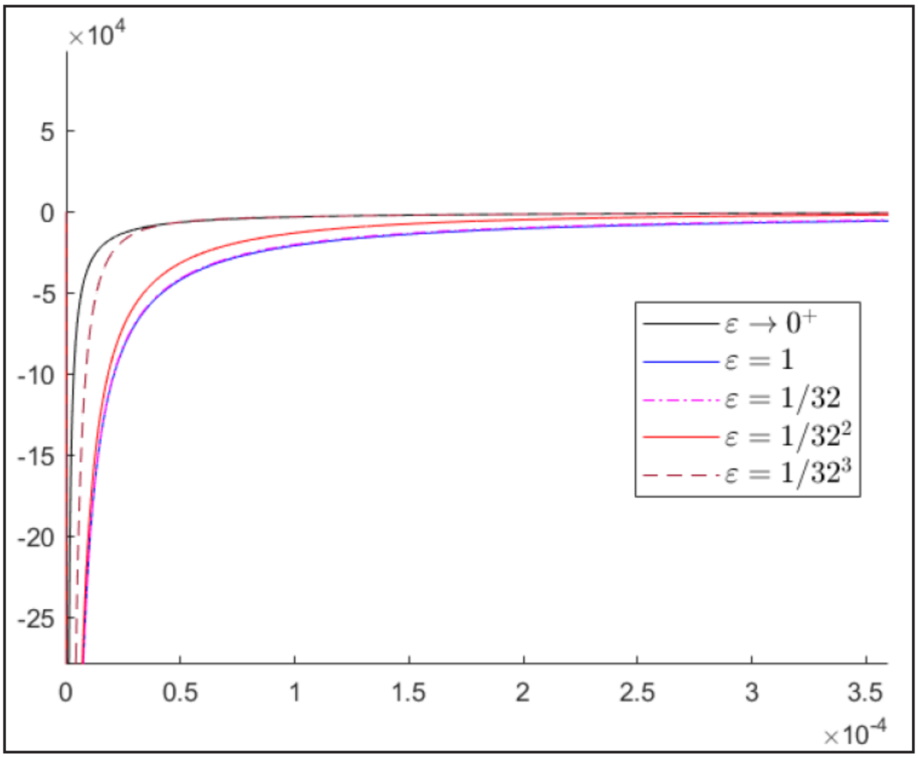
whose solution, according to the theory of second-order final-value problems (Teschl, 2012) exists, is unique and is given by

$$\phi_0(\rho) = -\frac{\rho^2}{6} - \frac{1}{3\rho} + \frac{3}{2} \tag{24}$$

Figure 3 shows the solution (24) of the homogenized EEG forward problem (23) obtained via TSM in comparison with the exact solution of the original EEG forward problem (21) for decreasing values of ε. The convergence of $\phi\varepsilon(\rho)$ to $\phi 0(\rho)$ as $\varepsilon \to 0+(\rho)$ is clearly observed.

It is important to observe that, while the computational cost involved was not measured, it is evident that it is computationally cheaper to obtain the solution for a problem with constant and smooth coefficients as the homogenized problem (23) than that of problem with rapidly oscillating coefficients as problem 21. This measure is relevant for more complex applications of the TSM.

Figure 3 – Solution (24) the homogenized EEG problem (23) compared to the exact solution of the EEG forward problem (21) for decreasing values of ε



Source: from the authors

5 CONCLUSIONS

In this work, the two-space nonperiodic asymptotic homogenization method was applied to a radially symmetric 1D realization of the EEG forward problem. The calculation of the effective coefficient and related magnitudes was formalized and proved via a proposition. Also, the convergence of the solution of the original problem to the asymptotic (homogenized) solution obtained from the two-space method was illustrated. This work constitutes a small contribution on the two-space method, but it showed its potential for applications in EEG modeling, and more realistic models will be object of study in future works.

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