CIÊNCIA e NATURA

ISSN 2179-460X
OPEN ACCESS

Ci. e Nat., Santa Maria, v. 46, e73978, 2024• https://doi.org/10.5902/2179460X73978
Submitted: 01/31/23 • Approved: 12/21/23 • Published: 12/19/24

Mathematics

A note on a fractional Cournot-type model

Uma nota sobre um modelo de Cournot fracionário

Matheus Madeira Corrêa [I], **Adriano De Cezaro** [I],
Fabiana Travessini De Cezaro [I]

[I]Institute of Mathematicas, Statistics and Physics, Federal University of Rio Grande, RS, Brazil

ABSTRACT

In this contribution, we analyze the memory effect associated with fractional derivatives in a Cournot-type oligopoly model. Such an analysis will be made from the comparison of the solutions of a duopoly with fractional derivatives of different orders. We show that the firm that has more memory on the competition, given by the fractional derivative, will have advantages over its competitor in the long run. Moreover, we show that the choices of fractional derivatives $\alpha_j \in (0, 3/5)$ for both firms $j = 1, 2$ will result in advantages in profit compared to the profit of firms without memory.

Keywords: Cournot-type model; Oligopolies; Memory; Fractional derivatives

RESUMO

Nesta contribuição, analisamos o efeito de memória associado a derivadas fracionárias em um modelo de oligopólio do tipo Cournot. Tal análise será feita a partir da comparação das soluções de um duopólio com derivadas fracionárias de diferentes ordens. Mostramos que a firma que tiver mais memória sobre a concorrência, dada pela derivada fracionária, terá vantagens sobre sua concorrente no longo prazo. Além disso, mostramos que as escolhas de derivadas fracionárias $\alpha_j \in (0, 3/5)$ para ambas as firmas $j = 1, 2$ resultarão em vantagens no lucro quando comparado com o lucro das firmas sem memória.

Palavras-chave: Modelo do tipo Cournot; Oligopólios; Memória; Derivadas fracionárias

1 INTRODUCTION

In the market structure, the oligopolies are those in which the market is dominated by a small number of firms seeking to maximize profits, for example Dixon (2001); Tirole (1988); Varian (2006). In particular, Cournot competition is an economic model, named after Antoine Augustin Cournot Dixon (2001); Tirole (1988); Varian (2006), used to describe an oligopoly in which firms compete on the amount of output they will produce. Cournot's model assumes that all firms are economically rational and act strategically, usually seeking to maximize profit given their competitors' decisions, which they decide on independently of each other at the same time, do not cooperate, produce heterogeneous products, and each firm's output decision affects the good's price Dixon (2001); Tirole (1988); Varian (2006).

A fundamental assumption of Cournot models is that each firm aims to maximize its profit, based on the expectation that its own output decisions will not have an effect on the decisions of its rivals. Furthermore, the market price is set so that demand equals the total amount $Q = \sum_{i=1}^{N} q_i$ produced by all oligopolistic firms N. As a result, each firm $i \in \{1, \cdots, N\}$ takes the produced quantity q_j, for which $j \neq i$ is set by its competitors as a given, evaluates its residual demand, and then behaves as a monopoly.

The mathematical models for oligopolies, including the Cournot competition model, are modeled in terms of economic marginal indicators (as marginal prices and marginal costs), which are derived using integer order derivatives Rosser (2022); Tarasov (2019). As a consequence, the corresponding model predicts that the competition of firms will completely disregard the history of market competition in a process of pure amnesia Tarasov (2019); Tarasova and Tarasov (2017,1).

Today, it is reasonable to assume that decision-making takes into account a competitive background, or "memory". Hence, generalizing economic indicators using fractional derivatives Tarasov (2019); Tarasova and Tarasov (2017,1) is an alternative to modeling competition in oligopoly markets.

Paper organization and novelties: In Section 2, we briefly revisit the notions of fractional calculus that will be used in these contributions at Subsection 2.1 as well as the Cournot-type model for duopoly in Subsection 2.2. In Section 2 we present the main novelty of this contribution. It starts with the derivation of a generalized firms' competition in a Cournot-type problem, using fractional Riemann-Liouville derivatives Diethelm (2010) as a generalized marginal indicator, e.g. Tarasov (2019); Tarasova and Tarasov (2017,1). This marginal indicator can be interpreted as a weighted average of historical profits of the firms involved (direct result of the definition of fractional derivatives, Diethelm (2010)). In Section 3.1, we show that the firm that has more memory of the long-term competitive process is able to obtain a strategy with a higher relative profit, that is, the firm that uses the fractional derivative of lower order. Section 4 is devoted to some concluding remarks and pointing out some further directions of investigation.

2 MATERIALS AND METHODS

For the sake of completeness, we will briefly present in this section the derivatives and integrals of fractional order that will be used throughout this work. For a complete overview of fractional derivatives, see Diethelm (2010). Subsequently, we will present in an equally objective way the Cournot-type model from which we will derive the generalized marginal indicators. Such calculations enable the main conclusion of this contribution: In the long run, the firm with a better memory of competitive history will earn a higher relative profit.

2.1 Fractional calculus

The fractional calculus had its origin in 1695, in a letter written by L'Hôpital to Leibniz, where he questions the interpretation of $\frac{d^n y}{dx^x}$ when $n = \frac{1}{2}$. Leibniz's response was the following expression: $D^{\frac{1}{2}}Y(x) := x\sqrt{dx : x}$ adding that *"This is an apparent paradox from which important applications will be obtained*". Since then, many mathematicians have been studying fractional calculus, its properties, and its applications. In what follows, we will briefly present the definitions of integrals and fractional derivatives in order to leave the work self-contained. For more details, see Diethelm (2010) and references.

The fractional integral or order $\beta \in [0, 1)$, according to Riemann-Liouville, is defined by

$$J_t^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-x)^{\beta-1} f(x)dx \,, \qquad t > a \,, \tag{1}$$

Where

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt \tag{2}$$

is the Gamma function. In particular, for $\beta = 1$ we have the traditional integral of integer order.

The fractional derivative is defined in several ways Diethelm (2010). In what follows, we chose to use the Riemann-Liouville fractional derivative, whose fractional derivative of order $1-\alpha$, for $\alpha \in [0,1]$, is given by

$$D^{1-\alpha} f(x) = \frac{d}{dx}\left[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{-\alpha} f(t)dt\right] . \tag{3}$$

Note that if $\alpha = 0$ the fractional derivative is exactly the traditional integer derivative. It is worth mentioning that $1-\alpha$ is the level of memory given by the fractional derivative operator $D^{1-\alpha}$. The property of fractional derivative memory on a dynamics can be observed in (Diethelm, 2010, Remark 6.4).

2.2 The Cournot-type model for a duopoly

In this contribution, we will consider two firms operating in the market (a duopoly or an oligopoly with $N = 2$ firms). We also assume that the inverse demand function is given by

$$P(q_1, q_2) = a - bQ \, , \tag{4}$$

where a and b are constants such that $a > b$, $q_1 = q_1(t)$ and $q_2 = q_2(t)$ represent the output of firms one and two respectively, with $Q = q_1 + q_2$ representing total production and $P(q_1, q_2)$ representing the product price function.

We will also assume that the production cost of each unit is constant and equal to c, so that the cost function of the firm $i \in \{1, 2\}$ is defined by

$$C_i = cq_i \, . \tag{5}$$

In this way, profits $\Pi_i(q_i, q_j)$ of firms $i \in \{1,2\}$ and $i \neq j$, $j = 1,2$ can be written, respectively, as

$$\begin{cases} \Pi_1(q_1,q_2) = q_1 P(q_1,q_2) - q_1 c \\ \Pi_2(q_1,q_2) = q_2 P(q_1,q_2) - q_2 c\,. \end{cases} \tag{6}$$

Substituting (5) and (4), with $Q = q_1 + q_2$, in (6) we get the Cournot-type model

$$\begin{cases} \Pi_1(q_1,q_2) = aq_1 - bq_1^2 - bq_2 q_1 - q_1 c \\ \Pi_2(q_1,q_2) = aq_2 - bq_2^2 - bq_2 q_1 - q_2 c \end{cases} \tag{7}$$

3 A FRACTIONAL ORDER COURNOT-TYPE APPROACH FOR A DUOPOLY

The next step is to introduce the first novelty in the analysis of the problem; that is, to find the equilibrium points when we derive the equilibrium equations with relation to the fractional derivatives. The economical interpretation of this generalized marginal quantity was obtained in Tarasov (2019); Tarasova and Tarasov (2017,1). In other words, we will partially derive the profit functions Π_1 and Π_2 given in (7) using fractional derivatives as defined in (3) of order $1-\alpha_1$ and $1-\alpha_2$, $\alpha_1, \alpha_2 \in [0,1)$, respectively. From (3), it follows that

$$\begin{cases} D^{1-\alpha_1}\Pi_{1,q_1}(q_1,q_2) = (a - bq_2 - c)\frac{q_1^{\alpha_1}}{\Gamma(1+\alpha_1)} - b\frac{2q_1^{1+\alpha_1}}{\Gamma(2+\alpha_1)} \\ D^{1-\alpha_2}\Pi_{2,q_2}(q_1,q_2) = (a - bq_1 - c)\frac{q_2^{\alpha_2}}{\Gamma(1+\alpha_2)} - b\frac{2q_2^{1+\alpha_2}}{\Gamma(2+\alpha_2)}\,, \end{cases} \tag{8}$$

where $D^{1-\alpha_1}\Pi_{1,q_1}$ and $D^{1-\alpha_2}\Pi_{2,q_2}(q_1,q_2)$ represent the fractional partial derivative of Π_i and with respect to q_i, for $i = 1,2$ respectively.

Given the properties of fractional derivatives, for example, Diethelm (2010), it follows that the possible critical points, which in this case we will call ”fractional equilibrium points”, are given by equations $D^{1-\alpha_1}\Pi_{1,q_1}(q_1,q_2) = 0$ and $D^{1-\alpha_2}\Pi_{2,q_2}(q_1,q_2) = 0$ in (8). Therefore, it follows from (8) that the ”fractional equilibrium points” satisfy

$$\begin{cases} q_1 = \frac{(1+\alpha_1)(a-c)}{2b} - \frac{(1+\alpha_1)q_2}{2} \\ q_2 = \frac{(1+\alpha_2)(a-c)}{2b} - \frac{(1+\alpha_2)q_1}{2}\,. \end{cases} \tag{9}$$

It is worth noting that if $\alpha_1 = \alpha_2 = 0$ in (9), we have the Cournot equilibrium point, which is well known in the literature; for example, see Tirole (1988); Varian (2006).

Following the ideas of Snyder et al. (2013), let us assume that the quantities produced $q_i = q_i(t)$, for $i = 1, 2$, respectively, are continuously differentiable with respect to time. As a result, we can write $-q_i(t + \Delta t)$ in both sides of each equation of the system (9), use Taylor's theorem with Lagrange remainder and divided by Δt to deduces that

$$q_i^{\cdot}(t) = q_i(t) - q_i(t + \Delta t) + o(\Delta t)\,, \quad i = 1, 2, \tag{10}$$

where $q_i^{\cdot}$ denotes the ordinary derivative of the quantity q_i with respect to time.

Taking $\Delta t \to 0$ in (9) and (10), we obtain the following linear system of ordinary differential equations

$$\begin{cases} q_1^{\cdot}(t) = & \frac{(1+\alpha_1)(a-c)}{2b} - \frac{(1+\alpha_1)}{2} q_2(t) - q_1(t) \\ q_2^{\cdot}(t) = & \frac{(1+\alpha_2)(a-c)}{2b} - \frac{(1+\alpha_2)}{2} q_1(t) - q_2(t)\,, \end{cases} \tag{11}$$

whose general solution is given by

$$\begin{array}{rl} q_1(t) = & k_1 e^{\lambda_1 t} v_1 \;+\; k_2 e^{\lambda_2 t} w_1 \;+\; p_1 \\ q_2(t) = & k_1 e^{\lambda_1 t} v_2 \;+\; k_2 e^{\lambda_2 t} w_2 \;+\; p_2\,, \end{array} \tag{12}$$

where $\lambda_1 = -\frac{\sqrt{(1+\alpha_1)(1+\alpha_2)}+2}{2}$, $\lambda_2 = \frac{\sqrt{(1+\alpha_1)(1+\alpha_2)}-2}{2}$ are the eigenvalues, $v_1 = w_1 = 1$ $v_2 = \frac{\sqrt{(1+\alpha_1)(1+\alpha_2)}}{(1+\alpha_1)} = -w_2$, are the corresponding eigenvector coordinates, $p_1 = \frac{(\alpha_1+1)(\alpha_2-1)(c-a)}{((\alpha_1+1)\alpha_2+(\alpha_1-3))b}$, $p_2 = \frac{((\alpha_1-1)(\alpha_2+1))(c-a)}{((\alpha_1+1)\alpha_2+(\alpha_1-3))b}$ and k_j are general constant, determined by the initial conditions $(q_1(t=0), q_2(t=0))$.

It follows from the substitution of (12) in (7) that the difference $\Pi(t) := \Pi_2(q_1(t), q_2(t)) - \Pi_1(q_1(t), q_2(t))$, satisfies

$$\begin{aligned}\Pi(t) = k_1 &\left[e^{\lambda_1 t}\overbrace{((a-c)(v_2-v_1) - b(p_1+p_2)(v_2-v_1) - b(v_1+v_2)(p_2-p_1))}^{c_1}\right] \\ &+ k_1^2\left[e^{2\lambda_1 t}\overbrace{(-b(w_1+w_2)(v_2-v_1))}^{c_2}\right] \\ &+ k_1k_2\left[e^{(\lambda_1+\lambda_2)t}\overbrace{(-b(w_1+w_2)(v_2-v_1) - b(v_1+v_2)(w_2-w_1))}^{c_3}\right] \\ &+ k_2\left[e^{\lambda_2 t}\overbrace{((a-c)(w_2-w_1) - b(p_1+p_2)(w_2-w_1) - b(w_1+w_2)(p_2-p_1))}^{c_4}\right] \\ &+ k_2^2\left[e^{2\lambda_2 t}\overbrace{(-b(w_1+w_2)(w_2-w_1))}^{c_5}\right] \\ &+ \overbrace{(a-c)(p_2-p_1) - b(p_1+p_2)(p_2-p_1)}^{c_6}\,.\end{aligned} \tag{13}$$

In particular,

$$c_6 = -\frac{\overbrace{2(a-c)^2}^{>0}}{b}(\alpha_2-\alpha_1)\left[\frac{\left(\overbrace{(1-\alpha_2)(1-\alpha_1)}^{>0}\right)}{((\alpha_1+1)\alpha_2+(\alpha_1-3))^2}\right] \tag{14}$$

3.1 On the long-term average strategy

Our next step is to show that in a long-term strategy, the firm with the most memory power has a winning strategy on average. It is worth mentioning that the level of memory of the firm i is given by $1-\alpha_i$, for $i = 1, 2$.

Let $\Pi_i(t)$, for $i = 1, 2$ be the price function of firms 1 and 2, respectively. Also consider the parameters in the equation (12), with $\alpha_1, \alpha_2 \in [0.1[$. Suppose further that $b \geq 2/3$. In a long-term average strategy (T large enough), while in $\alpha_2 < \alpha_1$, firm 2's strategy is the winner. Otherwise, firm 1's strategy wins.

Proof. From (13) we have that, for T large enough

$$\int_0^T \Pi(t)dt = -k_1 \frac{2}{-\sqrt{(1+\alpha)(1\beta)}+2} c_1 - k_1^2 \frac{1}{-\sqrt{(1+\alpha)(1\beta)}+2} c_2 - k_1 k_2 \frac{1}{-2} c_3 - k_2 \frac{1}{\sqrt{(1+\alpha)(1\beta)}-2} c_4 - k_2^2 \frac{1}{\sqrt{(1+\alpha)(1\beta)}-2} c_5 + Tc_6 \,. \tag{15}$$

It follows from (15) and the fact that $\lambda_1 = -\frac{\sqrt{(1+\alpha_1)(1+\alpha_2)}+2}{2} < 0$, $\lambda_2 = \frac{\sqrt{(1+\alpha_1)(1+\alpha_2)}-2}{2} < 0$, since $\alpha_1, \alpha_2 \in [0,1]$, that, for T large enough, the signal of $\int_0^T \Pi(t)dt$ is determined by the signal of c_6.

The hypotheses that $\alpha_1, \alpha_2 \in [0, 1[$ and (14) imply that the signal of c_6 is given by the signal of

$$\alpha_2 - \alpha_1 \,, \tag{16}$$

from which the assertion follows. □

As we saw above, the firm that uses more memory has a higher relative profit in the long run than its competitor. However, it still remains to understand how such profit relates to the profit of the firm itself if it does not use memory (with integer derivatives). Given the symmetry of the analyzed model, the results obtained for firm 1 are analogous to firm 2 which is then omitted.

Let

$$\Pi_1(t) := \Pi_1|_{\alpha_1=0,\alpha_2=0} - \Pi_1|_{\alpha_1,\alpha_2=0} \,,$$

the difference between the profit of firm 1, when both firms assume the derivatives of integer order $\Pi_1|_{\alpha_1=0,\alpha 2=0}$ and $\Pi_1|_{\alpha_1,\alpha_2=0}$ the one with firm 1 assuming the fractional derivative of order $\alpha \in (0,1)$ and firm 2 the integer order, respectively.

Arguing analogous to the derivation of (12), we can see that all coefficients are exponential because $\lambda_1 < 0$ and $\lambda_2 < 0$, except

$$\hat{c}_6 = \frac{(a-c)^2}{b} \left[\frac{(\alpha_1 - 3)^2 - 9(1-\alpha_1^2)}{9(\alpha_1-3)^2} \right] = \frac{(a-c)^2}{b} \left[\frac{\alpha_1(10\alpha_1 - 6)}{9(\alpha_1-3)^2} \right] . \tag{17}$$

Therefore, it follows from arguments analogous to (15) and (17) that the average profit of $\Pi_1(t)$ will be positive whenever $\alpha_1 \in (0, \frac{3}{5})$. In other words, the firm 1 with fractional derivatives increases its profit compared to its one profit with integer derivatives in the Cournot model as far as $\alpha_1 \in (0, \frac{3}{5})$. The conclusions for firm 2 follow similarly.

In conclusion, according to the calculations presented here, there is a dominant strategy for choosing the order of the derivative that will be used to solve the fractional Cournot problem. This strategy consists of choosing fractional derivatives of order $1-\alpha_i$ where $\alpha_i \in (0, \frac{3}{5})$ for $i = 1, 2$. In other words, $(1-\alpha_1, 1-\alpha_2)$ with $\alpha_1, \alpha_2 \in (0, \frac{3}{5})$ is the best strategy for both companies if T is large enough.

4 CONCLUSIONS AND FURTHER DIRECTIONS

In this work, we introduce fractional-order derivatives in the study of competition between firms in a Cournot model. We show that with this strategy, the average profit is favorable for the firm that uses more memory in the long-term competitive process. Furthermore, comparing the average profits of firms with and without memory (with fractional and integer derivatives), we show that the choice of $\alpha_1, \alpha_2 \in (0, \frac{3}{5})$ is the best strategy for both firms if T is sufficiently large.

The results obtained here can be useful in the sense of showing that there is a vast horizon to be explored at the intersection of fractional calculus theory and economic theory. We believe that what was done in this work can serve as a motivation for future investigation, with data extracted from real situations, to determine whether fractional derivatives can be efficient in solving time-dependent economic problems. However, there are still a variety of open questions that deserve attention. Some examples are: What is the effect of applying fractional derivatives in models with more firms? What is to be expected from fractional derivatives in a model where the firms' products are differentiated? Or, what if one results on applying fractional derivatives in a model where firms adopt different strategies? Such questions will be investigated in future contributions.

ACKNOWLEDGEMENTS

This research is grateful for the partial support from FAPERGS - Research Foundation of the State of Rio Grande do Sul (Protocol: 47179.584.21167.16042020), and to PPGMC-FURG.

REFERENCES

Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type.* Springer Science Business Media, UK.

Dixon, H. D. (2001). *Surfing Economics: Essays for the Inquiring Economist*. Springer.

Rosser, J. (2022). *The Development of Complex Oligopoly Dynamics Theory. In: Puu, T., Sushko, I. (eds) Oligopoly Dynamics.* Springer, Berlin, Heidelberg.

Snyder, B. C., Van Gorder, R. A., & Vajravelu, K. (2013). Continuous-time dynamic games for the cournot adjustment process for competing oligopolists. *Applied Mathematics and Computation*, 219(12):6400–6409.

Tarasov, V. E. (2019). On history of mathematical economics: Application of fractional calculus. *Mathematics*, 6(7):498–509.

Tarasova, V. & Tarasov, V. E. (2017). Economic interpretation of fractional derivatives. *Progr. Fract. Differ. Appl.*, 3(1):1–6.

Tarasova, V. & Tarasov, V. E. (2018). Concept of dynamic memory in economics. *Communications in Nonlinear Science and Numerical Simulation*, 55(1):127–145.

Tirole, J. (1988). *The Theory of Industrial Organization*. MIT Press, UK.

Varian, H. R. (2006). *Intermediate microeconomics: a modern approach*. W. W. Norton & Company, USA.

Author contributions

1 – Matheus Madeira Corrêa
Mathematician
https://orcid.org/0009-0002-9704-5370 • matheusmolyna@hotmail.com
Contribution: Numerical implementation, Writing, Review

2 – Adriano De Cezaro (Corresponding Author)
Mathematician
https://orcid.org/0000-0001-8431-9120 • decezaromtm@gmail.com
Contribution: Writing, Review

3 – Fabiana Travessini De Cezaro
Teacher, Mathematician
https://orcid.org/0000-0001-9401-5315 • fabi.travessini@gmail.com
Contribution: Writing, Review

How to cite this article

Corrêa, M. M., De Cezaro, A., & De Cezaro, F. T. (2024). A note on a fractional Cournot-type model. *Ciência e Natura*, Santa Maria, v.46, e73978. DOI: https://doi.org/10.5902/2179460X73978.