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Special Edition

Asymptotic homogenization with finite elements for an orthotropic radially microperiodic sphere

Homogeneização assintótica com elementos finitos para uma esfera ortotrópica radialmente microperiódica

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ABSTRACT

This paper proposes a semi-analytical methodology that combines the asymptotic homogenization method (AHM) with the finite elements method (FEM) to solve boundary-value problems with rapidly oscillating coefficients. This approach is motivated by the convergence difficulties observed when this type of problem is addressed directly via FEM, whereas the AHM has shown to be efficacious for obtaining good generic approximations of the exact solution. Illustratively, this AHM-FEM methodology is developed for the mechanical equilibrium problem of a radially microperiodic orthotropic sphere under hydrostatic pressure, which allows its validation by comparing with the AHM analytical solution. Specifically, the effective coefficients and the homogenized and local problems are calculated via AHM, and then their analytical and FEM solutions are obtained. Finally, to validate the semianalytical methodology, the generic solutions are applied in an example and, from the obtained results, a comparison is made between the analytical AHM solution and the semi-analytical AHM-FEM solution .

Keywords: Rapidly Oscillating Coefficients; Asymptotic Homogenization; Finite Elements; Microperiodic Sphere

RESUMO

Este trabalho propõe uma metodologia semianalítica que combina os métodos de homogeneização assintótica (MHA) e os elementos finitos (MEF) para a resolução de problemas de valores de contorno com coeficientes rapidamente oscilantes. Essa proposta está fundamentada na dificuldade de convergência que a aplicação direta do MEF apresenta para esse tipo de problema, enquanto o MHA mostra-se eficaz para produzir boas aproximações genéricas da solução exata. Ilustrativamente, a metodologia MHA-MEF é desenvolvida para o problema do equilíbrio mecânico de uma esfera ortotrópica radialmente microperiódica sob pressão hidrostática, o qual permite sua validação através da comparação com a resolução analítica via MHA. Especificamente, são obtidos os coeficientes efetivos e os problemas homogeneizado e local através do MHA e, em seguida, encontram-se as soluções genéricas desses problemas, tanto analiticamente quanto pelo MEF. Por fim, para que a metodologia semianalítica seja validada, as soluções genéricas obtidas são aplicadas em um exemplo e, a partir dos resultados obtidos, é feita a comparação entre as soluções obtidas analiticamente pelo MHA e semi-analiticamente via MHA-MEF.

Palavras-chave: Coeficientes rapidamente oscilantes; Homogeneização assintótica; Elementos finitos; Esfera microperiódica

1 INTRODUCTION

Materials with rapidly oscillating physical properties, either natural or man-made, are widely present in everyday life (Leitzke, 2017). In engineering, the micro-heterogeneity can be seen, for example, in porous and composite materials. The latter consist of the combination of different constitutive materials for obtaining a desired physical behavior which the constituents do not exhibit (Sampaio and Rocha, 2010).

The finite elements method (FEM) is a numerical method commonly used to analyze structures in the field of structural engineering. However, when applied directly to problems with rapidly oscillating coefficients that model micro-heterogeneous materials, this approach requires a large number of elements to discretize the structure, which is computationally expensive and compromises its convergence (Lima *et al*., 2021).

On the other hand, the asymptotic homogenization method (AHM – see, as example, Bakhavalov and Panasenko, 1989) is an efficient tool to homogenize problems with rapidly oscillating coefficients, that is, to produce problems with constant coefficients whose solutions are good approximations of the solutions of the original problems with rapidly oscillating coefficients. In addition to the homogenized problems, local problems and effective coefficients are obtained. Specifically, the AHM proposes a series solution in powers of the small geometric parameter ε representing separation of structural scales, $\varepsilon = l/L \ll 1$, where l and L are the characteristic lengths of the microscopic and macroscopic scales, respectively. The microscopic scale is the scale in which the heterogeneity occurs and is noticeable and, in order to homogenize the problem, it is necessary that the material satisfy the continuum hypothesis in this scale, so the hypothesis of equivalent homogeneity is valid and effective properties can be attributed to the material (Leitzke, 2017; Lima, 2016).

The aim of this paper is to present a semianalytical methodology combining AHM and FEM to solve boundary-value problems with rapidly oscillating coefficients. The AHM-FEM methodology is applied to a problem of a radially microperiodic orthotropic sphere under hydrostatic pressure, which allows a comparison with the analytical solution via AHM (for the particular case of a uniform sphere, see: Sampaio, 2009). Using the AHM, the local and homogenized problems and the effective coefficients are obtained. These problems are generically solved analytically and via FEM. To validated the FEM formulation, the generic solutions are applied to an example with a cosine microperiodicity, which is an adaptation of the porosity distribution model proposed by Chen *et al*. (2015), and the numerical results are compared with the analytical ones.

2 METHODOLOGY

2.1 Problem formulation

Let ε>0 be a small parameter. Consider an orthotropic sphere (Ting, 1999) with unit radius and whose elastic properties are ε-periodic in the radial direction and otherwise constant. The center of the sphere coincides with the origin of the spherical coordinate system. The mechanical equilibrium of the sphere, which is under hydrostatic pressure, is modeled by the equilibrium equation

$$\frac{d\sigma_{\rho\rho}^{\varepsilon}}{d\rho}+\frac{2}{\rho}\left(\sigma_{\rho\rho}^{\varepsilon}-\sigma_{\theta\theta}^{\varepsilon}\right)=0,\ \ \rho\in(0,1) \tag{1}$$

the constitutive relations

$$\begin{aligned}\sigma_{\rho\rho}^{\varepsilon}&=C_{11}^{\varepsilon}(\rho)\frac{du^{\varepsilon}}{d\rho}+2C_{12}^{\varepsilon}(\rho)\frac{u^{\varepsilon}}{\rho},\\ \sigma_{\theta\theta}^{\varepsilon}&=C_{12}^{\varepsilon}(\rho)\frac{du^{\varepsilon}}{d\rho}+\left(C_{22}^{\varepsilon}(\rho)+C_{23}^{\varepsilon}(\rho)\right)\frac{u^{\varepsilon}}{\rho},\end{aligned} \tag{2}$$

and the boundary conditions $u^{\varepsilon}(0)=0$ and $\sigma_{\rho\rho}^{\varepsilon}(1)=-p$, where: $\sigma_{\rho\rho}^{\varepsilon}$ and $\sigma_{\theta\theta}^{\varepsilon}$ are the stresses in the radius ρ and angle θ directions; C_{11}^{ε}, C_{12}^{ε} , C_{22}^{ε} and C_{23}^{ε} are continuously differentiable and ε-periodic functions of the elastic properties with $C_{11}^{\varepsilon}>\frac{2(C_{12}^{\varepsilon})^2}{C_{22}^{\varepsilon}+C_{23}^{\varepsilon}}>0$ and $C_{22}^{\varepsilon}>|C_{23}^{\varepsilon}|>0$; u^{ε} is the radial displacement; and ρ>0 is an constant pressure.

Substituting Eq. (2) into Eq. (1), the equilibrium equation is rewritten as

$$\frac{d}{d\rho}\left[a^{\varepsilon}(\rho)\ \frac{du^{\varepsilon}}{d\rho}\right]\ +\ b^{\varepsilon}(\rho)\ \frac{du^{\varepsilon}}{d\rho}\ +\ c^{\varepsilon}(\rho)\ u^{\varepsilon}=0,\ \ \rho\in(0,1), \tag{3}$$

and, being $y=\frac{\rho}{\varepsilon}$ and the local and global variables, respectively, the coefficients are

$$\begin{aligned}a^{\varepsilon}(\rho)&=a(y)=C_{11}(y)=C_{11}^{\varepsilon}(\rho),\\ b^{\varepsilon}(\rho)&=b(\rho,y)=\frac{2}{\rho}a(y)=\frac{2}{\rho}C_{11}(y)=\frac{2}{\rho}C_{11}^{\varepsilon}(\rho),\\ c^{\varepsilon}(\rho)&=c(\rho,y)=\ \frac{2}{\rho}\left[\frac{d}{d\rho}\left(C_{12}^{\varepsilon}(\rho)\right)+\frac{1}{\rho}\left(C_{12}^{\varepsilon}(\rho)-C_{22}^{\varepsilon}(\rho)-C_{23}^{\varepsilon}(\rho)\right)\right]\\ &=\frac{2}{\rho}\frac{d}{d\rho}\left(C_{12}^{\varepsilon}(y)\right)+\frac{2}{\rho^2}\left(C_{12}^{\varepsilon}(\rho)-C_{22}^{\varepsilon}(\rho)-C_{23}^{\varepsilon}(\rho)\right)\\ &=\varepsilon^{-1}\frac{2}{\rho}\frac{\partial}{\partial y}\left(C_{12}(y)\right)+\frac{2}{\rho^2}\left(C_{12}(y)-C_{22}(y)-C_{23}(y)\right)'\\ &=\varepsilon^{-1}\frac{\partial}{\partial y}\left(d(\rho,y)\right)+\frac{1}{\rho}d(\rho,y)+e(\rho,y)\\ d^{\varepsilon}(\rho)&=d(\rho,y)=\frac{2}{\rho}C_{12}(y)=\frac{2}{\rho}C_{12}^{\varepsilon}(\rho),\\ e^{\varepsilon}(\rho)&=e(\rho,y)=-\frac{2}{\rho^2}\left(C_{22}(y)+C_{23}(y)\right)=-\frac{2}{\rho^2}\left(C_{22}^{\varepsilon}(\rho)+C_{23}^{\varepsilon}(\rho)\right).\end{aligned} \tag{4}$$

Notice that the chain rule $\frac{d}{d\rho}=\frac{\partial}{\partial\rho}+\frac{1}{\varepsilon}\frac{\partial}{\partial y}$ was applied in defining $c^{\varepsilon}(\rho)=c(\rho,y)$. In addition, observe that $a(y)$, $b(\rho,y)$, $c(\rho,y)$, $d(\rho,y)$ and $e(\rho,y)$ are 1-periodic in y. Furthermore, note that, due to the similarity between the equilibrium equations and the constitutive relations of the sphere, of the disk, and of the cylinder, it is possible to rewrite, generically, their equilibrium equations as Eq. (3), changing only the definition of the coefficients , $a^{\varepsilon}(\rho)$, $b^{\varepsilon}(\rho)$ and $c^{\varepsilon}(\rho)$ corresponding to each case.

Considering Eqs. (2) and (4), the displacement boundary conditions are

$$u^{\varepsilon}(0) = 0, \left[a^{\varepsilon}(\rho)\,\frac{du^{\varepsilon}}{d\rho} + d^{\varepsilon}(\rho)\,u^{\varepsilon}(\rho)\right]_{\rho=1} = -p. \tag{5}$$

2.2 AHM application

A formal asymptotic solution (FAS) of the problem defined by Eqs. (3)-(5) is sought as the following asymptotic expansion of the exact solution $u^{\varepsilon}(\rho)$:

$$u^{(2)}(\rho,\varepsilon) = u_0(\rho,y) + \varepsilon u_1(\rho,y) + \varepsilon^2 u_2(\rho,y), \tag{6}$$

where unknown functions $u_k(\rho,y)$, $\mathrm{k} = 0,1,2$, are twice continually differentiable in both variables and 1-periodic in y. By substituting Eq. (6) into Eq. (3), applying the chain rule and rearranging by powers of ε, it is obtained that

$$\begin{aligned}
&\varepsilon^{-2}\left[\frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial y}\right)\right] + \varepsilon^{-1}\left[\frac{\partial}{\partial \rho}\left(a(y)\frac{\partial u_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial \rho}\right) + \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1}{\partial y}\right)\right.\\
&\qquad \left. + b(\rho,y)\frac{\partial u_0}{\partial y} + \frac{\partial}{\partial y}\left(d(\rho,y)\right)u_0(\rho,y)\right] + \varepsilon^{0}\left[\frac{\partial}{\partial \rho}\left(a(y)\frac{\partial u_0}{\partial \rho}\right) + \frac{\partial}{\partial \rho}\left(a(y)\frac{\partial u_1}{\partial y}\right),\right.\\
&\qquad + \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1}{\partial \rho}\right) + \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_2}{\partial y}\right) + b(\rho,y)\frac{\partial u_0}{\partial \rho} + b(\rho,y)\frac{\partial u_1}{\partial y}\\
&\qquad \left. + \frac{\partial}{\partial y}\left(d(\rho,y)\right)u_1(\rho,y) + d(\rho,y)\frac{u_0(\rho,y)}{\rho} + e(\rho,y)u_0(\rho,y)\right] = \mathcal{O}(\varepsilon)
\end{aligned} \tag{7}$$

where $O(\varepsilon)$ gathers the terms with positive powers of ε.

From Eq. (7), and considering as a small parameter, it is concluded that $u^{(2)}$, given by Eq. (6), to be considered a FAS, the coefficients of nonpositive powers of ε must be null, which produces the following recurrence of equations for obtaining functions u_k, $\mathrm{k} = 0,1,2$:

$$\varepsilon^{-2} : \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial y}\right) = 0, \tag{8}$$

$$\varepsilon^{-1} : \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1}{\partial y}\right) = -\frac{\partial}{\partial \rho}\left(a(y)\frac{\partial u_0}{\partial y}\right) - \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial \rho}\right) - b(\rho,y)\frac{\partial u_0}{\partial y} - \frac{\partial}{\partial y}\big(d(\rho,y)\big)u_0(\rho,y), \tag{9}$$

$$\begin{aligned}\varepsilon^{0} : \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_2}{\partial y}\right) = & -\frac{\partial}{\partial \rho}\left(a(y)\frac{\partial u_0}{\partial \rho}\right) - \frac{\partial}{\partial \rho}\left(a(y)\frac{\partial u_1}{\partial y}\right) \\ & -\frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1}{\partial \rho}\right) - b(\rho,y)\frac{\partial u_0}{\partial \rho} - b(\rho,y)\frac{\partial u_1}{\partial y} \\ & -\frac{\partial}{\partial y}\big(d(\rho,y)\big)u_1(\rho,y) - \frac{1}{\rho}d(\rho,y)u_0(\rho,y) + e(\rho,y)u_0(\rho,y)\end{aligned}. \tag{10}$$

By applying Eq. (6) into Eqs. (5), the boundary conditions that functions $u_k(\rho,y)$, $k = 0,1,2$ must satisfy are obtained:

$$\begin{aligned}& u_0(0{,}0) = u_1(0{,}0) = u_2(0{,}0) = 0,\ \left[a(y)\frac{\partial u_0}{\partial y}\right]_{(\rho,y)=(1,0)} = 0, \\ & \left[a(y)\frac{\partial u_0}{\partial \rho} + d(\rho,y)u_0(\rho,y) + a(y)\frac{\partial u_1}{\partial y}\right]_{(\rho,y)=(1,0)} = -p.\end{aligned} \tag{11}$$

Existence and uniqueness of 1-periodic solutions of Eqs. (8)-(10) are guaranteed by the following *Lemma* (Bakhavalov and Panasenko, 1989). Consider the equation $\frac{d}{dy}\left(a(y)\frac{dN}{dy}\right) = F(y)$, where data $F(y)$ and $\alpha(y)$ are 1-periodic and differentiable, being $\alpha(y)$ strictly positive and bounded. A necessary and sufficient condition for a 1-periodic solution $N(y)$ to exist is that $\langle F(y)\rangle = 0$, where $\langle\cdot\rangle \equiv \int_0^1(\cdot)dy$ is the mean value operator. In addition, the solution $N(y)$ is unique up to an additive constant, that is, $N(y,C) = \widetilde{N}(y) + C$, where $\widetilde{N}(0) = 0$ and C is a constant.

Applying the Lemma to Eq. (8), it can be concluded that u_0 depends only on the global variable ρ. So, the right-hand side of Eq. (9) is rewritten as $-\frac{da}{dy}\frac{du_0}{d\rho} - \frac{\partial d}{\partial y}u_0(\rho)$ and, by applying the Lemma considering that $\left\langle\frac{da}{dy}\right\rangle = \left\langle\frac{\partial d}{\partial y}\right\rangle = 0$ due to the 1-periodicity in y of $\alpha(y)$ and $d(\rho,y)$, it follows that the solution u_1 exists and is 1-periodic in y. Also, the format of the right-hand side suggests to seek for u_1 as

$$u_1(\rho,y) = N_{11}(y)\frac{du_0}{d\rho} + N_{12}(\rho,y)u_0(\rho), \tag{12}$$

where the so-called local functions $N_{11}(y)$ and $N_{12}(\rho,y)$ are, respectively, the solutions, 1-periodic in y, of the so-called local problems given by

$$\frac{d}{dy}\left(a(y)\,\frac{dN_{11}}{dy}\right) = -\frac{da}{dy},\;\; N_{11}(0) = 0, \tag{13}$$

$$\frac{\partial}{\partial y}\left(a(y)\,\frac{\partial N_{12}}{\partial y}\right) = -\frac{\partial d}{\partial y},\;\; N_{12}(0{,}0) = 0, \tag{14}$$

and whose existence and uniqueness are guaranteed by the Lemma. So, $N_{11}(y)$ and $N_{12}(\rho,y)$ are

$$N_{11}(y) = \int_0^y \left(\frac{\hat{a}}{a(y)} - 1\right) ds,\; N_{12}(\rho, y) = \left\langle\frac{d(\rho,y)}{a(y)}\right\rangle (N_{11}(y) + y) - \int_0^y \frac{d(\rho,s)}{a(s)}\, ds, \tag{15}$$

where, considering Eqs. (13) and (15), the so-called effective coefficient $\hat{a}$ is given by

$$\hat{a} = \left\langle a(y) + a(y)\frac{dN_{11}}{dy}\right\rangle = a(y) + a(y)\frac{dN_{11}}{dy} = \left\langle \left(a(y)\right)^{-1}\right\rangle^{-1}. \tag{16}$$

On the other hand, notice that $\left\langle\frac{d}{dy}\left(a(y)N_{11}(y)\right)\right\rangle = 0$, as both $\alpha(y)$ and $N_{11}(y)$ are 1-periodic. This is taken into account when the Lemma is applied to Eq. (10) after substituting Eq. (12) into it and recalling that u_0 depends only on the global variable ρ. As a consequence, in order to guarantee the existence of a solution u_2, 1-periodic in y, there must exist the solution u_0 of the so-called homogenized equation

$$\hat{a}\frac{d^2u_0}{d\rho^2} + \hat{b}(\rho)\frac{du_0}{d\rho} + \hat{c}(\rho)u_0(\rho) = \; 0, \tag{17}$$

where $\hat{b}(\rho)$ and $\hat{c}\,(\rho)$ are effective coefficients given by

$$\hat{b}(\rho) = \frac{2\hat{a}}{\rho},\;\; \hat{c}(\rho) = \left\langle\frac{d(\rho,y)}{a(y)}\right\rangle\left(\frac{1}{2}\hat{b}(\rho) - \hat{a}\left\langle\frac{d(\rho,y)}{a(y)}\right\rangle\right) + \left\langle\frac{\left(d(\rho,y)\right)^2}{a(y)}\right\rangle - \langle e(\rho,y)\rangle \equiv \frac{2\hat{k}}{\rho^2}, \tag{18}$$

where $\hat{k} > 0$ is a constant. Finally, considering Eq. (11), the so-called homogenized problem is defined by Eq. (17) and the boundary conditions

$$u_0(0) = 0, \quad \hat{a}\left.\frac{\partial u_0}{\partial \rho}\right|_{\rho=1} + d(1,0)u_0(1) = -p. \tag{19}$$

With such considerations, the first two terms of Eq. (6), taking Eq. (12) into account, the following FAS is obtained:

$$u^{(1)}(\rho,\varepsilon) = u_0(\rho) + \varepsilon\left(N_{11}\left(\frac{\rho}{\varepsilon}\right)\frac{du_0}{d\rho} + N_{12}\left(\rho,\frac{\rho}{\varepsilon}\right)u_0(\rho)\right). \tag{20}$$

2.3 Analytical solution of the homogenized problem

Observe that the homogenized equation, Eq. (17), with the effective coefficients defined by Eqs. (16) and (18), can be written as a Cauchy-Euler equation

$$\rho^2\frac{d^2u_0}{d\rho^2} + 2\rho\frac{du_0}{d\rho} + \frac{2\hat{k}}{\hat{a}}u_0(\rho) = 0, \tag{21}$$

whose general solution is

$$u_0(\rho) = C_1\rho^{\frac{-1+\sqrt{1-\frac{8\hat{k}}{\hat{a}}}}{2}} + C_2\rho^{\frac{-1-\sqrt{1-\frac{8\hat{k}}{\hat{a}}}}{2}}, \tag{22}$$

where constants C_1 and C_2 are found by applying the boundary conditions in Eqs. (19) to Eq. (22), so it follows that

$$C_1 = -\frac{2p}{\hat{a}\left(-1+\sqrt{1-\frac{8\hat{k}}{\hat{a}}}\right)+2d(1,0)}, \quad C_2 = 0, \tag{23}$$

which, substituted into Eq. (22), produce the solution of the homogenized problem:

$$u_0(\rho) = -\frac{2p}{\hat{a}\left(-1+\sqrt{1-\frac{8\hat{k}}{\hat{a}}}\right)+2d(1,0)}\rho^{\frac{-1+\sqrt{1-\frac{8\hat{k}}{\hat{a}}}}{2}}. \tag{24}$$

2.4 Solutions of the homogenized and local problems using FEM

The weak formulation of the homogenized problem, Eqs. (17) and (19), is

$$-w(1)[p + d(1,0)u_0(1)] - \int_0^1 \hat{a}\rho^2 \frac{du_0}{d\rho}\frac{dw}{d\rho}\, d\rho + \int_0^1 2\hat{k}u_0(\rho)w(\rho)\, d\rho = 0, \tag{25}$$

being $w(\rho)$ a general weight function, which is homogeneous in the essential boundary conditions. By applying the FEM to Eq. (25), the following system of linear equations is obtained:

$$[K]\{D\} = \{F\}, \tag{26}$$

where

$$[K] = \sum_{e=1}^{m}\left(\left[L^{(e)}\right]^T\left[k^{(e)}\right]\left[L^{(e)}\right]\right) - \left[L^{(m)}\right]^T\{\phi(1)\}d(1,0)\{\phi(1)\}^T\left[L^{(m)}\right], \tag{27}$$

$$\{F\} = \left[L^{(m)}\right]^T\{\phi(1)\}p, \tag{28}$$

$$\left[k^{(e)}\right] = \int_{\rho_i^{(e)}}^{\rho_f^{(e)}}\left(\{\phi(\rho)\}\, 2\hat{k}\, \{\phi(\rho)\}^T - \left\{\frac{d\phi}{d\rho}\right\}\hat{a}\rho^2\left\{\frac{d\phi}{d\rho}\right\}^T\right)d\rho, \tag{29}$$

being $\{D\}$ the nodal displacements vector, $\{\phi\}$ is the basis function vector and $[L^{(e)}]$ is the association matrix between the local nodal parameters to the global ones.

Also, the weak formulation of the local problem for $N_{11}(y)$, Eq. (13), is

$$\int_0^1 \frac{da}{dy}w(y)\, dy - \int_0^1 a(y)\frac{dN_{11}}{dy}\frac{dw}{dy}dy = 0, \tag{30}$$

being $w(y)$ a weight function, which is homogeneous in the essential boundary conditions. By applying FEM to Eq. (30), the following system of linear equations is obtained:

$$[K]\{V\} = [F], \tag{31}$$

where

$$[K] = \sum_{e=1}^{m} \left[L^{(e)}\right]^T [k^{(e)}] \left[L^{(e)}\right] \tag{32}$$

$$\{F\} = \sum_{e=1}^{m} \left[L^{(e)}\right]^T \{f\}, \tag{33}$$

$$\left[k^{(e)}\right] = \int_{y_i^{(e)}}^{y_f^{(e)}} \left\{\frac{d\phi}{dy}\right\} a(y) \left\{\frac{d\phi}{dy}\right\}^T dy, \tag{34}$$

$$[K] = \sum_{e=1}^{m} \left[L^{(e)}\right]^T [k^{(e)}] \left[L^{(e)}\right], \tag{35}$$

being $\{V\}$ a vector with nodal values of $N_{11}(y)$. The essential boundary conditions of the problem, considering the 1-periodicity of $N_{11}(y)$, are $N_{11}(0) = N_{11}(1) = 0$. In this paper, taking into consideration that the objective is to present and investigate the efficacy of the AHM-FEM semianalytical methodology, only two-node elements are used, both in local and homogenized problems. Others types of elements will be studied in future works in order to improve the efficiency of the proposed AHM-FEM methodology.

3 RESULTS

InordertovalidatetheAHM-FEMmethodologypresentedinthispaper,considerthat the elastic property varies 1-periodically as the law $\alpha(y)$, $A_1[1 - a_0 \cos(2\pi y)]$, $a_0 = 1 - \frac{A_0}{A_1}$, which is an adaptation of a porosity distribution model in Chen *et al.* (2015). The corresponding effective coefficient is $\hat{a} = A_1\sqrt{1 - a_0^2}$. The other elastic properties vary according to the relations $d(\rho, y) = \eta b(\rho, y)$ and $(\eta + \gamma)b(\rho, y) + \rho e(\rho, y) = 0$, adapted from Sampaio (2009), so that $\hat{k} = \eta(2\eta - 1)(A_1 - \hat{a}) - \gamma A_1 < 0$. Thus, it follows form Eq. (15) that the solutions of local problems, Eqs. (13) and (14), obey the relation $\rho N_{12}(\rho, y) = 2\eta N_{11}(y)$, with

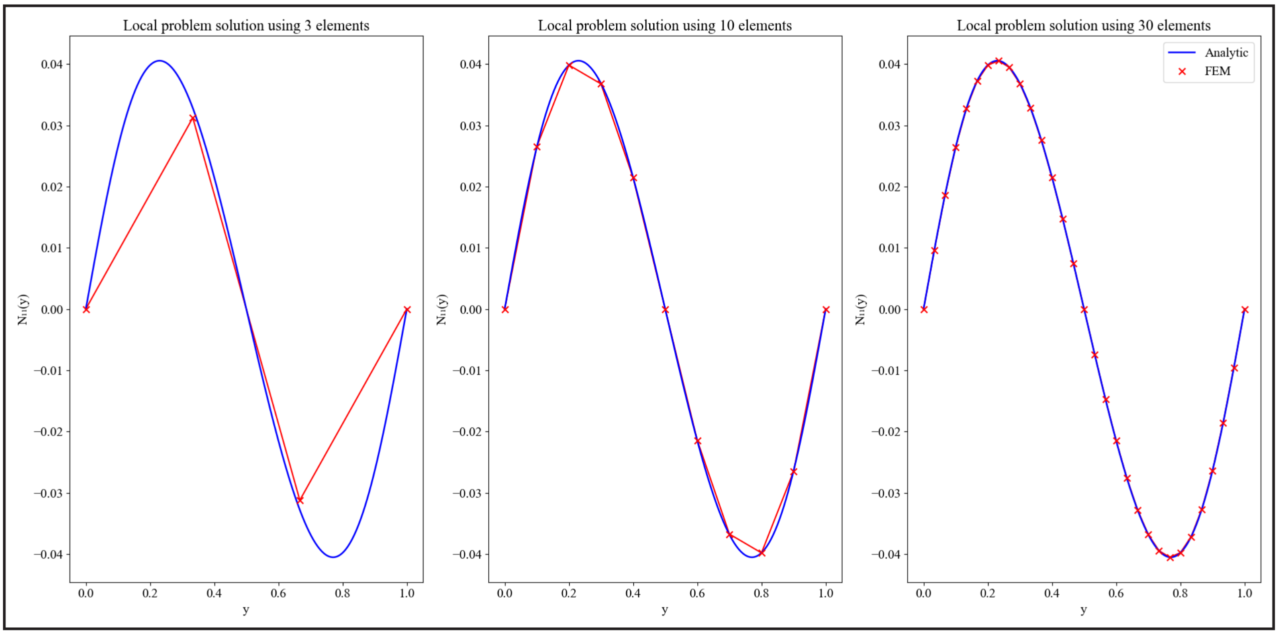
$$N_{11}(y) = \frac{1}{\pi}\arctan\left(\frac{A_1}{\hat{a}}(1 + a_0)\tan \pi y\right) - y + H\left(y - \frac{1}{2}\right),\ y \in [0,1], \tag{36}$$

where $H(\cdot)$ is the Heaviside function.

Here, the values of the parameters employed in the computational experiments are the following: $A_0 = 150$ and $A_1 = 200$ (Chen *et al.*, 2015), and η=0,5 and γ=0,055 (Sampaio, 2009), and p=1.

The analytical expression in Eq. (36) of the local solution $N_{11}(y)$ of the problem in Eq. (13) is presented in Fig. 1 together with its numerical approximation via FEM for 3, 10 and 30 elements, respectively. As expected, it can be observed that increasing the number of elements improves the quality of the numerical approximation in comparison with the analytical solution.

Figure 1 – Local problem solutions

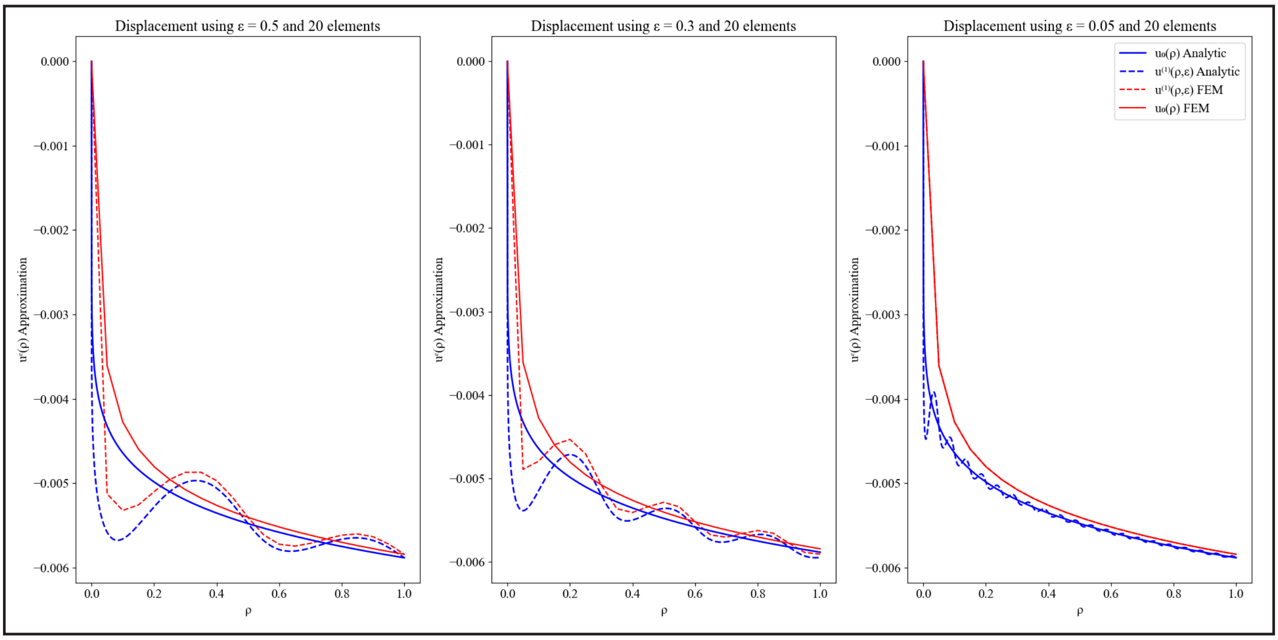


Source: Authors (August, 2022)

Caption: Comparison between the exact analytical expression of $N_{11}(y)$ in Eq. (36) and its numerical approximation via FEM for increasing number of elements.

On the other hand, the exact solution $u^{\varepsilon}(\rho)$ is approximated by solution $u_0(\rho)$ of the homogenized problem and by the FAS $u^{(1)}(\rho,\varepsilon)$, respectively, whose analytical AHM and semianalytical AHM-FEM realizations are presented in Figs. 2-4, where their behaviors with respect to the variation of parameter ε (Figs. 2 and 3) and to the number of elements (Fig. 4) are shown.

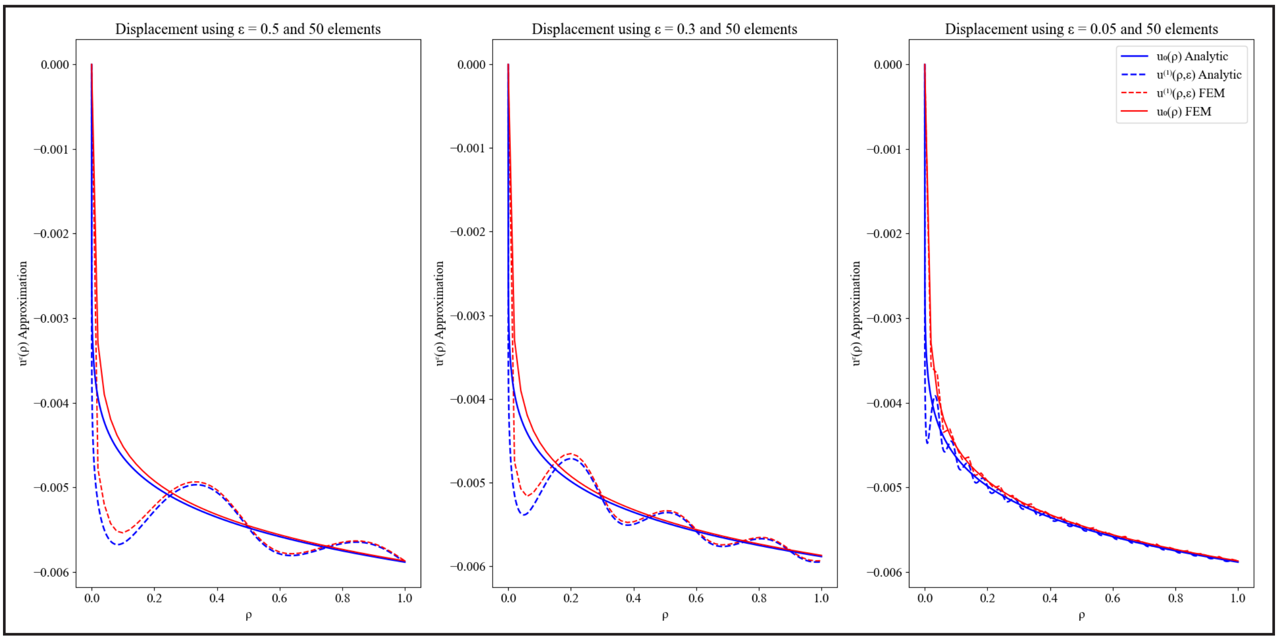
Figure 2 – Homogenized problem solutions using 20 elements



Source: Authors (August, 2022)

Caption: FEM approximations of the homogenized problem solution u_0 and of the FAS $u^{(1)}$ in comparison with their exact analytical solutions to a decreasing ε (20 elements).

Figure 3 – Homogenized problem solutions using 50 elements



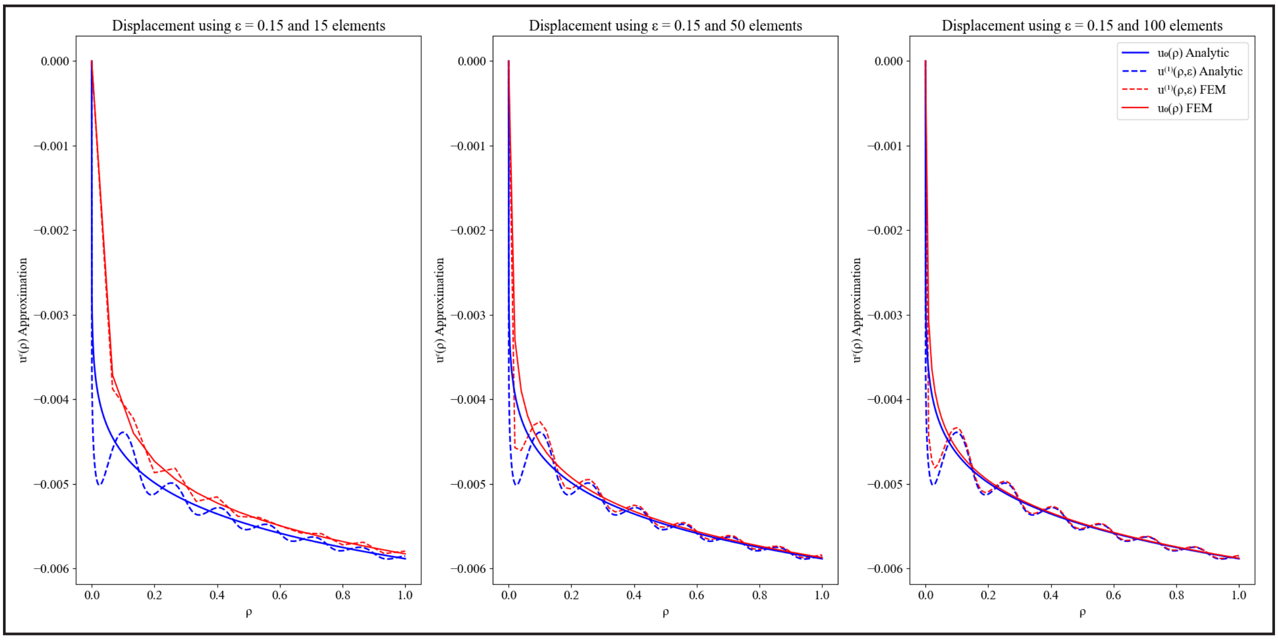
Source: Authors (August, 2022)

Caption: FEM approximations of the homogenized problem solution u_0 and of the FAS $u^{(1)}$ in comparison with their exact analytical solutions to a decreasing ε (50 elements).

In Figs. 2 and 3, notice that both the analytical AHM and the semianalytical AHM-FEM approaches, the FAS $u^{(1)}(\rho,\varepsilon)$ tends to the solution $u_0(\rho)$ of the homogenized problem as the value of ε decreases. However, for the two-node element used here, a small number of elements makes the FEM approximations of $u_0(\rho)$, $N_{11}(y)$ and $u^{(1)}(\rho,\varepsilon)$ not accurate in comparison with their analytical counterparts and, therefore, it is necessary to consider a larger number of elements.

So, as illustrated in Fig. 4, increasing the number of elements makes the FEM approximation tend to the corresponding exact analytical solution. In addition, for sufficiently small values of ε, the four approximations of the exact solution u^{ε} of the original problem (u_0 both exact analytical and via FEM, and $u^{(1)}$ with u_0 and N_1 both exact analytical and via FEM) become practically indistinguishable. On the other hand, it is important to notice that the application of polynomial elements of higher order would improve the results obtained by the AHM-FEM approach, which has shown to be efficient. The study to other types of elements will be addressed in future works.

Figure 4 – Homogenized problem solutions



Source: Authors (August, 2022)

Caption: FEM approximations of the homogenized problem solution u_0 and of the FAS $u^{(1)}$ in comparison with their exact analytical solutions for increasing number of elements.

4 CONCLUSIONS

For both local and homogenized problem, the FEM solution converges to the analytical solution as the number of elements increases. Also, it can be noted that the solution $u^{(1)}$ converges to the solution u_0 while ε is decreasing. It is important to highlight that the regions with sharper variations need more elements to achieve convergence, making the use of higher order elements, such as quadratic and cubic ones, an interesting topic to be analyzed. However, as the objective of this paper is to present the efficacy of the AHM-FEM methodology, the behavior of the solution using different types of elements will be addressed in future works.

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