A new characterization of simple $K_3$-groups using same-order type

Uma nova caracterização de $k_3$-grupos simples usando o mesmo tipo de ordem

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ABSTRACT

Let $G$ be a group, define an equivalence relation $\sim$ as below:

$$\forall g, h \in G, g \sim h \iff |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$ and denoted by $\alpha(G)$. And $G$ is said a $\alpha_n$-group if $|\alpha(G)| = n$. Let $\pi(G)$ be the set of prime divisors of the order of $G$. A simple group of $G$ is called a simple $K_n$-group if $|\pi(G)| = n$. We give a new characterization of simple $K_3$-groups using same-order type. Indeed we prove that a nonabelian simple group $G$ has same-order type $\{r, m, n, k, l\}$ if and only if $G \cong PSL(2,q)$, with $q = 7, 8$ or $9$. This result generalizes the main results in (4), (6) and (8). Moreover based on the main result in (8) we have the natural question: Let $S$ be a nonabelian simple $\alpha_n$-group and $G$ a $\alpha_n$-group such that $|S| = |G|$. Then $S \cong G$. In this paper with a counterexample we give a negative answer to this question.

Keywords: Element order; Same-order type; Characterization; Simple group; $K_n$-group simple

RESUMO

Seja $G$ um grupo, definimos como uma relação de equivalência $\sim$:

$$\forall g, h \in G, g \sim h \iff |g| = |h|$$

O tamanho do conjunto de classes de equivalência dado por essa relação é chamado de mesmo tipo de ordem de $G$ e denotado por $\alpha(G)$. $G$ é chamado de um $\alpha_n$-grupo se $|\alpha(G)| = n$. Seja $\pi(G)$ o conjunto dos divisores primos da ordem de $G$. Um grupo simples de ordem $G$ é chamado de $K_n$- grupos simples se $|\pi(G)| = n$. Caracterizamos esses $K_3$- grupos simples usando outros de mesma ordem. Na verdade nós provamos que um grupo não abeliano $G$ tem o mesmo tipo de ordem $\{r, m, n, k, l\}$, se e somente se, $G$
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$\cong PSL(2,q)$, com $q = 7, 8$ ou $9$. Este é um resultado generalizado e os principais resultados em (4), (6) e (8). Além disso, com base no resultado principal em (8) nós temos uma questionamento natural: Seja $S$ um grupo simples não abeliano $\alpha_n$-grupo e $G$ a $\alpha_n$-grupo de tal modo que $|S| = |G|$. Então $S \cong G$. Neste artigo, com um contra-exemplo, damos uma resposta negativa a essa pergunta.

**Palavras-chave:** Ordem dos elementos; Mesmo tipo de ordem; Caracterização; Grupo simples; $K_n$-grupos simples

# 1 INTRODUCTION

In this paper all the groups we consider are finite.

Let $G$ a group and $\pi(G)$ be the set of element orders of $G$. Let $t \in \pi(G)$ and $s_t$ be the number of elements of order $t$ in $G$. Let $nse(G) = \{s_t | t \in \pi(G)\}$ the set of sizes of elements with the same order in $G$. Some authors have studied the influence of $nse(G)$ on the structure of $G$ (see (1), (5), (8) and (9)). For instance R. Shen in (6) proved that $A_4 \cong PSL(2, 3), A_5 \cong PSL(2, 4) \cong PSL(2, 5)$ and $A_6 \cong PSL(2, 9)$ are uniquely determined by $nse(G)$. As a continuation in (4) was proved that if $G$ is a group such that $nse(G) = nse(PSL(2, q))$, where $q \in \{7, 8, 11, 13\}$, then $G \cong PSL(2, q)$. In (7) and (8) new characterizations of $A_5$ were given using $nse(A_5)$. The authors in (7) proved that $A_5$ is the only group such that $nse(A_5) = \{1, 15, 20, 24\}$ and the authors in (8) generalized that a nonabelian simple group $G$ has same-order type $\{r, m, n, k\}$ if and only if $G \cong A_5$ (see Th. 1.1 (8)).

Let $G$ be a group, in (8) was defined an equivalence relation $\sim$ as below:

$$\forall g, h \in G, g \sim h \iff |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$ and denoted by $\alpha(G)$. And $G$ is said a $\alpha_n$-group if $|\alpha(G)| = n$. Note that $\alpha(G)$ is equal to the set of sizes of elements with the same order in $G$, hence $|nse(G)| = |\alpha(G)|$.

We give a new characterization of $PSL(2,7)$, $PSL(2,8)$ and $PSL(2,9)$ using same-order type.

**THEOREM 1.1.** Let $G$ be a simple $K_3$-group with same-order type $\{r, m, n, k, l\}$. Then $G \cong PSL(2,7), PSL(2,8)$ or $PSL(2,9)$.

This result generalizes the main results in (4), (6) and (8). Combination the main
results in (4) and (6) with Theorem 1.1 we have the following result

**COROLLARY 1.2.** A simple $K_3$-group $G$ has same-order type \{r, m, n, k, l\} if and only if $G \cong \text{PSL}(2,7)$, \text{PSL}(2,8) or \text{PSL}(2,9)$.

We see easily that the only $\alpha_1$-groups are 1 and a cyclic group of order 2. In (6) R. Shen characterized $\alpha_2$-group as nilpotent groups and $\alpha_3$-group as solvable groups. Moreover Taghvasani-Zarrin (see Th. 1.1 in (8)) showed that the only nonabelian simple $\alpha_4$-group is the $A_5$. As noted in (4) and (8) finite groups $G$ cannot be determined by $nse(G)$. Indeed in 1987 Thompson gave a first example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = \text{PSL}(3,4) \rtimes C_2$ be the maximal subgroups of Mathieu group $M_{23}$.

Then $nse(G_1) = nse(G_2)$, but $G_1 \not\cong G_2$.

Motivated by the main result in (8) about a new characterization of $A_5$ using same-order type, we have the natural question.

**QUESTION 1.3.** Let $S$ be a nonabelian simple $\alpha_n$-group and $G$ a $\alpha_n$-group such that $|S| = |G|$. Then $S \cong G$.

We give a negative answer to this question in the last section.

## 2 PROOF OF THEOREM 1.1

We need of one preliminary result to prove the main Theorem. The following result is a property very interesting of simple groups (see Lemma 2.7 in (8)).

**LEMMA 2.1.** Let $G$ be a nonabelian simple group. Then there exist two odd prime divisors $p$ and $q$ of the order of $G$ such that $s_p = s_q$.

In fact if $G$ is a nonabelian simple group then there exist two odd prime divisors $p$ and $q$ of the order of $G$ such that \{1, s_p, s_q\} $\subseteq \alpha(G)$ (see Corollary 2.8 in (8)).

We are now ready to conclude the proof of main Theorem.

Proof of Theorem 1.1: As $G$ is a nonabelian simple group, it follows that $s_2 > 1$, w.l.g. $r = 1$ and $s_2 = m$. From Lemma there exist odd prime divisors $p$ and $q$ of the order of $G$ such that $n = s_p \neq s_q = k$, hence $\pi(G) = \{2, p, q\}$ because $G$ is a simple $K_3$-group. Therefore \{1, s_2, s_p, s_q\} $\subseteq \alpha(G) = \{r, m, n, k, l\}$. So there exist a divisor $t \notin \pi(G)$ of order of $G$
such that \( s_t = l \). It’s well known that the only nonabelian simple groups of order divisible by exactly three primes are the following eight groups: \( \text{PSL}(2,q) \), where \( q \in \{5, 7, 8, 9, 17\} \), \( \text{PSL}(3,3) \), \( \text{PSU}(3,3) \) and \( \text{PSU}(4,2) \), see Th. 1 and Th. 2 in (3). Now we arguing as in the proof of Th. 1.1 in (8) and a GAP check yields that \( |\alpha(\text{PSL}(2,7))| = 5 \), \( |\alpha(\text{PSL}(2,8))| = 5 \), \( |\alpha(\text{PSL}(2,9))| = 5 \) and all others groups are \( \alpha_5 \)-group with \( n \geq 6 \) (except \( A_5 \) since \( |\alpha(A_5)| = 4 \)). The result is follows.

### 3 A COUNTEREXAMPLE TO A QUESTION 1.3

Now we give a counterexample to the Question 1.3. Firstly we observed that by the main Theorem in (6), we have that \( \alpha(\text{PSL}(2,7)) \) is uniquely determined and we have that \( \alpha(\text{PSL}(2,7)) = \{1,21,56,42,48\} \) hence \( \text{PSL}(2,7) \) is a \( \alpha_5 \)-group. Let \( G = Q_8 \times (C_7 \rtimes C_3) \), where \( Q_8 \) is the quaternion group of order 8. As \( |G| = 168 \) and \( G \) is a soluble group then is sufficient to prove that \( |\alpha(G)| = 5 \). Indeed the only 2-Sylow subgroup \( Q_8 \) is a normal subgroup of \( G \) and using Sylow’s Theorem it follows that \( s_2 = 8 \). Note that a 7-Sylow subgroup of \( G \) is isomorphic to \( C_7 \), and is a normal subgroup of \( Q_8 \cdot C_7 \) and \( C_7 \rtimes C_3 \), hence the normalizer \( N \) of \( C_7 \) has same order of \( G \). Again from Sylow’s Theorem we have that \( C_7 \) is a normal subgroup of \( G \) and \( s_7 = 56 \).

As the number of 3-Sylow subgroup of \( G \) is 7, then \( s_3 = 14 \). The number of elements of \( G \) of order 2, 4 are respectively 1 and 6, hence \( s_2 = 1 \), \( s_4 = 6 \) and consequently \( s_6 = 14 \), \( s_{12} = 84 \), \( s_{14} = 6 \) and \( s_{28} = 36 \) (because of direct product in the structure of \( G \)). Therefore \( \alpha(G) = \{1,1,14,6,14,6,84,6,36\} \) and \( G \) is a \( \alpha_5 \)-group. Clearly \( |\text{PSL}(2,7)| = 168 = |G| \) but \( \text{PSL}(2,7) \not\cong G \).

We can obtain others groups \( G \) with the computational group theory system GAP (2): \( G = C_7 \times (Q_8 \rtimes C_3) \) or \( G = C_2 \times ((C_{14} \times C_2) \rtimes C_3) \). These groups are also counterexamples to the Question 1.3.
4 CONCLUSION

We give the new characterization of some simple groups using the same-order type. Also, we give a negative answer for a natural question. Our main result generalizes some known results. There is a natural interest in this theme. This result depends on classification of finite simple groups (CFSG).

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