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Physics

Detailed demonstrations of the relations between coordinate systems in the celestial sphere

Demonstrações detalhadas das relações entre sistemas de coordenadas na esfera celeste

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ABSTRACT

Spherical Geometry is the basis for what we know as Positional Astronomy. This is one of the oldest approaches to Astronomy as a science, being used by ancient Greeks and possibly by other people before that. Concerning a more formal current mathematical description, there are several types of coordinate systems with respect to the celestial sphere. Each system differs in the choice of its referential plan. Several connections between elements of different types of coordinates are well known once it is a well-developed field. However, the reader will not find in the literature complete mathematical proofs for such formulas. Thus, this article fills that gap. We present in this text several formulas relating the coordinates Zenital Distance, Hour Angle, Azimuth, Declination and Geographic Latitude and their mathematical proofs in detail, explaining all possible steps. Our main contribution is in the form of the presentation and deduction of classical results from Positional Astronomy.

Keywords: Positional astronomy, Spherical astronomy, Mathematical astronomy, Spherical geometry, Spherical trigonometry

RESUMO

A Geometria Esférica é a base para o que conhecemos como Astronomia de Posição, que é uma das abordagens mais antigas da Astronomia como ciência, sendo usada pelos gregos antigos e possivelmente por outras civilizações passadas. No que diz respeito a uma descrição matemática atual mais formal, existem vários tipos de sistemas de coordenadas em relação à esfera celeste. Cada sistema difere na escolha de seu plano referencial. Diversas conexões entre elementos de diferentes tipos de coordenadas são bem conhecidas por se tratar de uma área bem desenvolvida. No entanto, o leitor não encontrará na literatura demonstrações matemáticas completas para tais fórmulas. Assim, este artigo preenche essa lacuna. Apresentamos neste texto diversas fórmulas relacionando as coordenadas Distância Zenital, Ângulo Horário, Azimute, Declinação e Latitude Geográfica e suas demonstrações matemáticas em detalhes, explicando todos os passos possíveis. Nossa principal contribuição está na forma de apresentação e dedução de resultados clássicos da Astronomia de Posição.

Palavras-chave: Astronomia de posição, Astronomia esférica, Astronomia matemática, Geometria esférica, Trigonometria esférica

1 INTRODUCTION

Speculation about the Universe is believed to have been occurring since prehistoric times. The search for understanding what surrounds us has been constant for mankind. Egyptian, Babylonian and Greek mathematicians and astronomers struggled to understand the nature of the stars in the sky, their magnitudes, distances from Earth, motions, as well as the shape of our planet (CARITÁ, 2018; CHODOROVÁ; *et al*., 2017; ROZENFELD, 2012; BARBOSA, 2002).

Around 550 BC, the Greek mathematician Thales of Miletus introduced the foundations of Egyptian geometry and astronomy to Greece. Thales and Anaximander (610–546 BC) was the first to propose celestial models based on the motion of the stars. Among the ancient Greek mathematicians who observed and conjectured about the sphericity of the Earth and the motion of the stars, we can mention: Pythagoras (572–497 BC), Philolaus of Croton (470–390 BC), Aristotle (384–322 BC), Aristarchus of Samos (310–230 BC), Eratosthenes (276–194 BC), Hipparchus (160–125 BC) and Ptolemy (85–165 AD). These mathematicians were responsible for the first studies of the motion of the stars and the terrestrial sphericity, and also explained the Moon phases. These concepts were used until the Renaissance, in the 16th century. It is worth mentioning that, during the lifetime of these mathematicians, Mathematics was undergoing a formalization process, that is, it was beginning to be presented formally with axioms, lemmas and theorems.

Taking a more formal approach, we can mention the works of the mathematician Euclid (300 BC), in particular his “Elements”. We will highlight the Euclid’s fifth postulate and its consequences. This fifth postulate says “If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles". Mathematicians of antiquity and the Middle Ages (such as: Proclus and John Wallis) believed that this postulate was actually a theorem and tried to proof it, but were unsuccessful (BARBOSA, 2002; ARCARI, 2008). The beginning of the 19th century found geometrists still in search for a "proof" of the Euclid's fifth postulate, but without success. However, such attempts to prove the postulate culminated in the formulation of new geometries called non-Euclidean (CHODOROVÁ; *et al*., 2017). These geometries received this name because they are no longer explained by the fundamentals developed by Euclid.

Spherical Geometry (a non-Euclidean geometry) was essential for developing location geography and navigation (CHODOROVÁ; *et al*., 2017; SANTOS; *et al*., 2021). Interestingly, long before the non-Euclidean geometries were made official, the Greeks, in antiquity, already used Spherical Geometry intuitively to study Astronomy. It was this civilization that proposed the concept of celestial sphere, initiating the first studies in Positional Astronomy (also known as Spherical Astronomy), which mainly concerns the directions in which the celestial bodies are seen, without worrying about their distances (CARITÁ, 2018; SMART, 1977).

The idea of a celestial sphere goes back to the time of ancient Greek astronomers, who conjectured that the star-encrusted sky was a large sphere, in which the Earth would be enveloped (ROZENFELD, 2012). From this conjecture, the concept of the celestial sphere arose as an imaginary sphere, with a gigantic radius, rotating around an axis (which is the extension of the Earth's axis of rotation), in which the celestial bodies and stars appear to move (SMART, 1977). Spherical Geometry offers tools for the analytical study of a celestial sphere model. We can count on a few different coordinate systems for the celestial sphere. Such systems are well known and provide ways to study positions of stars and celestial bodies in the sphere (CARITÁ, 2018; SMART, 1977; MCNALLY, 1974). These coordinate systems can be related. There are some mathematical formulas that connect coordinates of different systems. These connections are extremely useful for Positional Astronomy, since they favor the study of the dynamics of the apparent motions of stars and celestial bodies. As a result, it is possible to make predictions of when a star will, for example, cross a specific point in the sky. Among other applications, such formulas also facilitate the predictions of rising and setting times of stars.

In this context, this article aims to present mathematical proofs (using spherical trigonometry) for some connections between elements of the hourly, equatorial and horizontal coordinate systems, demonstrations that are not found in bibliographic references, books or articles on this subject.

2 MATHEMATICAL BACKGROUND

The mathematics used in Positional Astronomy is the Spherical Geometry, in particular the trigonometry in spherical triangles. There are good texts on Spherical Geometry[1]. Thus, in this work, we do not intend to exhaust the possibilities of approaching this subject. We do intend to present some concepts, which, in addition to presenting this geometry to the reader, will support the development of the formulas of section 5, which connect some coordinates of different systems in the celestial sphere, which is the main purpose of this work.

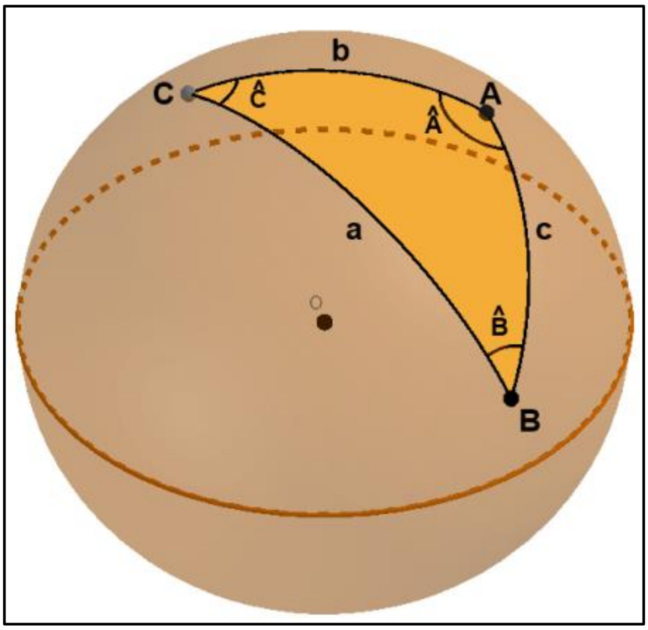
A surface where all of its points are equidistant from a fixed point is called a spherical surface. The fixed point is called the center and the distance from the center to any point on the surface is the radius. A sphere is a solid bounded by a spherical surface.

The boundary of every plane section of a sphere is a circle, called a great circle when the plane passes through the center of the sphere. We consider the

[1] The reader can check some in Smart (1977), Van Brummelen (2012), Murray (1900), Whittlesey (2019) and Santos (2020).

distance between two points on the spherical surface as the shorter arc of the great circle passing through them (this distance can naturally be measured as an angle).

Figure 1 - Example of a spherical triangle with vertices A, B and C, sides a, b and c, and internal angles $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ in a sphere of center O



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A spherical angle is the angle formed by two arcs of great circles with one end in common. The measure is the same as the dihedral angle of the planes of the arcs forming the angle. Three points not belonging to the same great circle, connected by the arcs of great circles that define their distances, determine a spherical triangle. Such points are called vertices. Figure 1 shows a spherical triangle with vertices A, B and C, sides measuring a, b and c, and internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$. In the illustrated layout, which we will adopt as a standard throughout our text, angle $\widehat{A}$ is opposite to side a, angle $\widehat{B}$ is opposite to side b and angle $\widehat{C}$ is opposite to side c. The sides and internal angles of a spherical triangle ABC always satisfy: $0° < a < 180°, 0° < b < 180°, 0° < c < 180°, 0° < \widehat{A} < 180°, 0° < \widehat{B} < 180°$ and $0° < \widehat{C} < 180°$.

Spherical triangles satisfy several interesting properties, such as (a) one side is always less than the sum of the other two and greater than their difference; (b) the sum of the internal angles is greater than 180° and less than 540°; and (c) the sum of the three sides is less than 360°. Concerning spherical trigonometry, we can highlight two main theorems: Law of Cosines and Law of Sines. Virtually all other known formulas that relate sides and internal angles to spherical triangles are deduced from these two theorems. In this work, we will prove these results as was proven in article Caritá (2018), but we will now use more details. In order to prove them, consider Figure 2, which illustrates a spherical triangle ABC in a sphere of center O.

Consider a straight line perpendicular to the OBC plane that passes through vertex A of the spherical triangle, P is the point of intersection of this line with the OBC plane. The line segment AP is perpendicular to the OBC plane. Using a similar process, starting from P, the PM and PN line segments are obtained, perpendicular to the OB and OC line segments, respectively. Thus, the Euclidean triangles APN, APM, ONP, OMP and OPA are right triangles. Therefore, it follows that

$$OA^2 = AP^2 + OP^2 \tag{1}$$

$$AM^2 = AP^2 + PM^2 \implies AP^2 = AM^2 - PM^2 \tag{2}$$

$$OP^2 = OM^2 + PM^2 \tag{3}$$

$$AN^2 = AP^2 + PN^2 \implies AP^2 = AN^2 - PN^2 \tag{4}$$

$$OP^2 = ON^2 + PN^2 \tag{5}$$

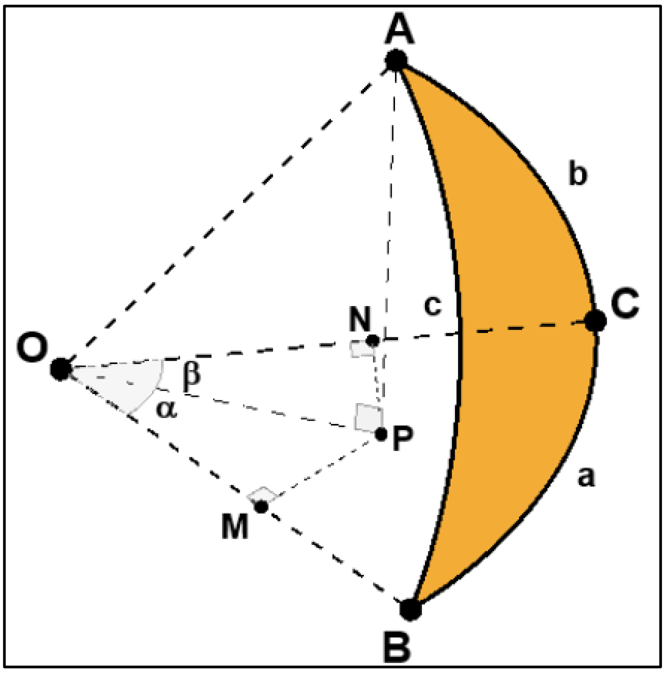
Replacing Equations (2) and (3) in (1), it follows:

$$OA^2 = AM^2 - PM^2 + OM^2 + PM^2 \implies OA^2 = AM^2 + OM^2$$

This implies that the Euclidean triangle OMA is also a right triangle. Replacing Equations (4) and (5) in (1), it follows:

$$OA^2 = AN^2 - PN^2 + ON^2 + PN^2 \implies OA^2 = AN^2 + ON^2$$

Figure 2 - Example of a spherical triangle and some useful projections. This figure shows the right triangles APN, APM, ONP, OMP and OPA, which are Euclidean. This means that useful formulas, known from traditional trigonometry (see Appendix A), can be used in proofs of Spherical Geometry theorems, as long as the projections shown in this image are used



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This implies that the Euclidean triangle ONA is also a right triangle. Therefore, all Euclidean triangles APN, APM, ONP, OMP, OPA, OMA and ONA are right triangles. From the Euclidean right triangle ONA, since the angle at vertex O coincides with side b of the spherical triangle, it follows that

$$\cos(b) = \frac{ON}{OA} \tag{6}$$

$$\sin(b) = \frac{AN}{OA} \tag{7}$$

From the Euclidean right triangle OMA, since the angle at vertex O coincides with side c of the spherical triangle, it follows:

$$\cos(c) = \frac{OM}{OA} \tag{8}$$

$$\sin(c) = \frac{AM}{OA} \tag{9}$$

From the Euclidean right triangle OMP, denoting by α the angle at the vertex in O, it follows that

$$\cos(\alpha) = \frac{OM}{OP} \tag{10}$$

$$\sin(\alpha) = \frac{MP}{OP} \tag{11}$$

From the Euclidean right triangle ONP, denoting by β the angle at the vertex in O, it follows that

$$\cos(\beta) = \frac{ON}{OP} \tag{12}$$

$$\sin(\beta) = \frac{NP}{OP} \tag{13}$$

Also from the Euclidean triangles APN and APM, it follows that

$$\cos(\widehat{N}) = \frac{NP}{AN}$$

$$\sin(\widehat{N}) = \frac{AP}{AN}$$

$$\cos(\widehat{M}) = \frac{MP}{AM}$$

$$\sin(\widehat{M}) = \frac{AP}{AM}$$

However, as $\widehat{M}$ and $\widehat{N}$ are the same dihedral angles that define the angles $\widehat{B}$ and $\widehat{C}$, respectively, it follow:

$$\cos(\widehat{C}) = \frac{NP}{AN} \tag{14}$$

$$\sin(\widehat{C}) = \frac{AP}{AN} \tag{15}$$

$$\cos(\widehat{B}) = \frac{MP}{AM} \tag{16}$$

$$\sin(\widehat{B}) = \frac{AP}{AM} \tag{17}$$

Since all the formulas mentioned above are related to Euclidean triangles, we can make use of formulas known from traditional trigonometry. Some useful formulas for this article are in Appendix A. From these reasonable previous considerations, we can enunciate and prove important results regarding trigonometry in spherical triangles, starting by the Law of Cosines and the Law of Sines.

Theorem 1 - (The Law of Cosines) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively. So:

$$\cos(a) = \cos(b).\cos(c) + \sin(b).\sin(c).\cos(\widehat{A})$$

$$\cos(b) = \cos(a).\cos(c) + \sin(a).\sin(c).\cos(\widehat{B})$$

$$\cos(c) = \cos(a).\cos(b) + \sin(a).\sin(b).\cos(\widehat{C})$$

Proof:

Once $\alpha + \beta = a$, from Equations (8) and (10) it follows that

$$OM = OA.\cos(c) \implies OP.\cos(\alpha) = OA.\cos(c) \implies$$

$$\implies OP.\cos(a-\beta) = OA.\cos(c)$$

Using the expression (37) from Appendix A combined with Equations (12) and (13):

$$OP\,(\cos(a).\cos(\beta) + \sin(a).\sin(\beta)) = OA.\cos(c) \implies$$

$$\implies OP\left(\cos(a)\frac{ON}{OP} + \sin(a)\frac{NP}{OP}\right) = OA.\cos(c)$$

Hence, from Equation (6), it follows that

$$OA.\cos(c) = OA.\cos(b).\cos(a) + NP.\sin(a) \tag{18}$$

From Equations (14) and (7) follows $NP = \cos(\widehat{C}).AN$ and $AN = \sin(b).OA$. Then

$$NP = \cos(\widehat{C}).\sin(b).OA$$

Replacing in Equation (18), it follows:

$$OA.\cos(c) = OA.\cos(b)\cos(a) + OA.\sin(b)\cos(\widehat{C})\sin(a)$$

Then

$$\cos(c) = \cos(a).\cos(b) + \sin(a).\sin(b).\cos(\widehat{C})$$

This proves one of the three proposed equalities. The other two are analogous.

Theorem 2 - (The Law of Sines) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively. So:

$$\frac{\sin(a)}{\sin(\widehat{A})} = \frac{\sin(b)}{\sin(\widehat{B})} = \frac{\sin(c)}{\sin(\widehat{C})}$$

Proof:

From Equations (9) and (17), it follows that

$$\sin(c) = \frac{AM}{OA} \implies AM = OA.\sin(c)$$

and

$$\sin(\widehat{B}) = \frac{AP}{AM} \implies AM = \frac{AP}{\sin(\widehat{B})}$$

where sin($\widehat{B}$)≠0, since 0°<$\widehat{B}$<180°. Thereby

$$OA.\sin(c) = \frac{AP}{\sin(\widehat{B})} \tag{19}$$

From Equations (7) and (15), it follows:

$$\sin(b) = \frac{AN}{OA} \implies AN = OA.\sin(b)$$

and

$$\sin(\widehat{C}) = \frac{AP}{AN} \implies AN = \frac{AP}{\sin(\widehat{C})}$$

where sin ($\widehat{C}$) ≠ 0, since 0° < $\widehat{C}$ < 180°. Thereby

$$OA.\sin(b) = \frac{AP}{\sin(\widehat{C})} \tag{20}$$

From Equations (19) and (20), we can conclude

$$\frac{AP}{OA} = \sin(c).\sin(\widehat{B}) = \sin(b).\sin(\widehat{C})$$

Which implies

$$\frac{\sin(b)}{\sin(\widehat{B})} = \frac{\sin(c)}{\sin(\widehat{C})}$$

This proves one of the equalities. The others are obtained in an analogous way.

Theorem 3 - (Law of Cosines - Part 2) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$,$\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively. So:

$$\sin(a).\cos(\widehat{C}) = \sin(b).\cos(c) - \cos(b).\sin(c).\cos(\widehat{A})$$
$$\sin(a).\cos(\widehat{B}) = \sin(c).\cos(b) - \cos(c).\sin(b).\cos(\widehat{A})$$
$$\sin(b).\cos(\widehat{A}) = \sin(c).\cos(a) - \cos(c).\sin(a).\cos(\widehat{B})$$
$$\sin(b).\cos(\widehat{C}) = \sin(a).\cos(c) - \cos(a).\sin(c).\cos(\widehat{B})$$
$$\sin(c).\cos(\widehat{B}) = \sin(a).\cos(b) - \cos(a).\sin(b).\cos(\widehat{C})$$
$$\sin(c).\cos(\widehat{A}) = \sin(b).\cos(a) - \cos(b).\sin(a).\cos(\widehat{C})$$

Proof:

Once $\alpha + \beta = a$, from Equations (11), it follows that

$$MP = OP.\sin(\alpha) = OP.\sin(a - \beta)$$

Using the expression (38) from Appendix A:

$$MP = OP.\sin(a).\cos(\beta) - OP.\cos(a).\sin(\beta) \tag{21}$$

From Equations (16) and (9) follows $MP = AM.\cos(\widehat{B})$ *and* $AM = OA.\sin(c)$, in other words:

$$MP = OA.\sin(c).\cos(\widehat{B}) \tag{22}$$

From Equations (12) and (6) follows $ON = OP.\cos(\beta)$ *and* $ON = OA.\cos(b)$, in other words:

$$OP.\cos(\beta) = OA.\cos(b) \tag{23}$$

From Equations (14) and (7) follows $NP = AN.\cos(\widehat{C})$ *and* $AN = OA.\sin(b)$, in other words:

$$NP = OA.\sin(b).\cos(\widehat{C})$$

Meantime from Equation (13), follows $NP = OP.\sin(\beta)$, so:

$$OP.\sin(\beta) = OA.\sin(b).\cos(\widehat{C}) \tag{24}$$

Replacing Equations (22), (23) and (24) in the Equation (21), it follows that

$$OA.\sin(c).\cos(\widehat{B}) = OA.\sin(a).\cos(b) - OA.\cos(a).\sin(b).\cos(\widehat{C})$$

Implying that

$$\sin(c).\cos(\widehat{B}) = \sin(a).\cos(b) - \cos(a).\sin(b).\cos(\widehat{C})$$

This proves one of the equalities. The others are obtained in an analogous way.

Theorem 4 - (Cotangent Formulas) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively. So:

$$\sin(a).\cot(c) = \sin(\widehat{B}).\cot(\widehat{C}) + \cos(a).\cos(\widehat{B})$$

$$\sin(a).\cot(b) = \sin(\widehat{C}).\cot(\widehat{B}) + \cos(a).\cos(\widehat{C})$$

$$\sin(b).\cot(a) = \sin(\widehat{C}).\cot(\widehat{A}) + \cos(b).\cos(\widehat{C})$$

$$\sin(b).\cot(c) = \sin(\widehat{A}).\cot(\widehat{C}) + \cos(b).\cos(\widehat{A})$$

$$\sin(c).\cot(a) = \sin(\widehat{B}).\cot(\widehat{A}) + \cos(c).\cos(\widehat{B})$$

$$\sin(c).\cot(b) = \sin(\widehat{A}).\cot(\widehat{B}) + \cos(c).\cos(\widehat{A})$$

Proof:

Starting from Theorem 3, consider the formula:

$$\sin(b).\cos(\widehat{C}) = \sin(a).\cos(c) - \cos(a).\sin(c).\cos(\widehat{B}) \tag{25}$$

Dividing the Equation (25) by $\sin(c)$:

$$\frac{\sin(b).\cos(\widehat{C})}{\sin(c)} = \frac{\sin(a).\cos(c)}{\sin(c)} - \frac{\cos(a).\sin(c).\cos(\widehat{B})}{\sin(c)} \tag{26}$$

$$\sin(a).\cot(c) = \frac{\sin(b).\cos(\widehat{C})}{\sin(c)} + \cos(a).\cos(\widehat{B})$$

From Theorem 2, it follows that

$$\frac{\sin(b)}{\sin(\widehat{B})} = \frac{\sin(c)}{\sin(\widehat{C})} \implies \sin(c) = \frac{\sin(b).\sin(\widehat{C})}{\sin(\widehat{B})} \quad (27)$$

Replacing the Equation (27) in the (26) it follows that

$$\sin(a).\cot(c) = \sin(\widehat{B}).\cot(\widehat{C}) + \cos(a).\cos(\widehat{B})$$

The other equations are shown analogously, starting from the other formulas of Theorem 3.

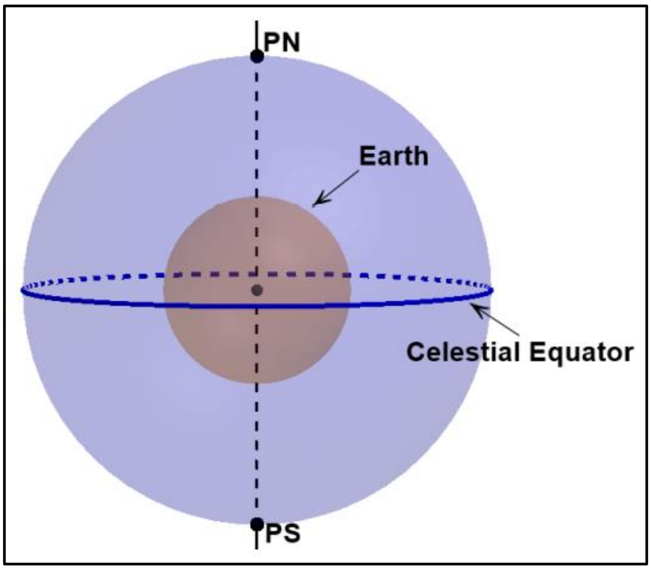
Now, once we have proved these important theorems of spherical geometry and trigonometry, we are able to investigate the relations between coordinate systems in the celestial sphere, which we now define and describe.

3 COORDINATE SYSTEMS IN THE CELESTIAL SPHERE

The idea of the celestial sphere comes from ancient Greek astronomers, who conjectured that the star-encrusted sky was a large sphere, in which the Earth would be enveloped. From this conjecture, the concept of the celestial sphere arose.

The celestial sphere is a gigantic imaginary sphere, centered on Earth, rotating around the extension of the Earth's axis of rotation, as shown in Figure 3. Despite the representation illustrated in Figure 3, it is important to mention that in this representation the Earth is dimensionless. The north celestial pole (PN) is the point at which the extension of the Earth's axis of rotation intersects the celestial sphere in the northern hemisphere. Similarly, the south celestial pole (PS) is the point at which the extension of the Earth's axis of rotation intersects the celestial sphere in the southern hemisphere. The celestial equator is the great circle in which the plane of the Earth's equator intersects the celestial sphere.

Figure 3 - The celestial sphere in blue and Earth in brown. The image also illustrates the two poles as well as the celestial equator

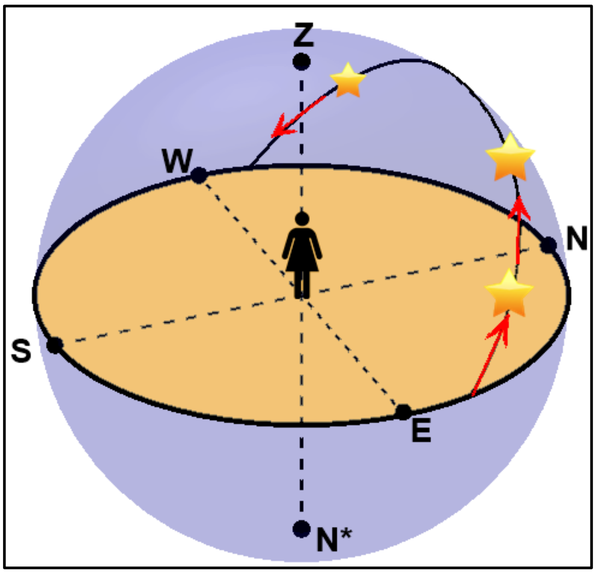


Source: Authors (2021)

Consider a straight line passing simultaneously through the center of the celestial sphere and the location of an observer on the Earth's surface. This line is called the observer's vertical (or vertical of the place). This line intersects the celestial sphere at two diametrically opposite points. These points are called Zenith (Z) and Nadir (N^*), the first is the point located above the observer and the second is the point located below. We define the observer's horizon as the plane tangent to the Earth and perpendicular to the aforementioned line, as shown in the Figure 4.

Due to the gigantic dimensions of the celestial sphere, the observer can also be considered its center. In this abstract concept of the celestial sphere, we can consider that the stars lie on the spherical surface. Depending on the observer's location on Earth, some stars can be seen crossing the horizon in an East (E) to West (W) motion. Other stars move in the same direction, but without ever crossing the horizon. This movement is called apparent motion. See Figure 4.

Figure 4 - The celestial sphere, the observer's horizon and the apparent motion of a star

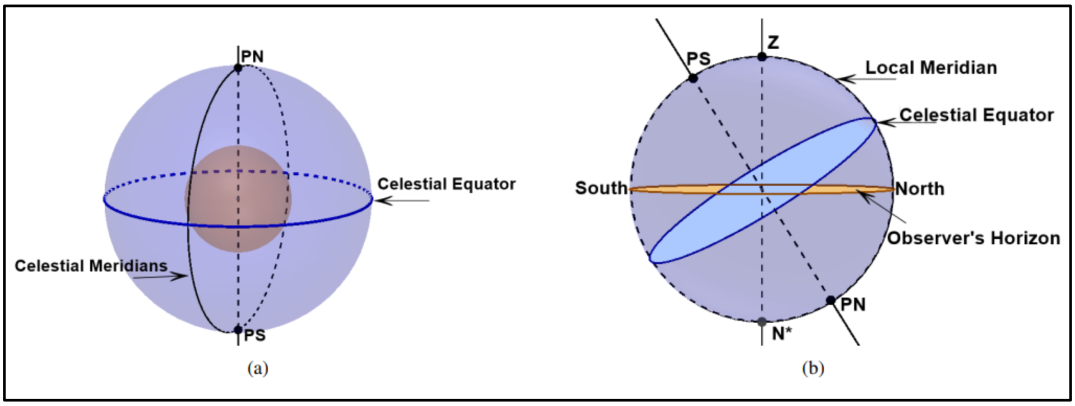


Source: Authors (2021)

In practice, for calculations, the radius of the celestial sphere is often considered unitary.

The great circles whose planes contain the two celestial poles are called celestial meridians, as shown in Figure 5 (a). The local meridian is the celestial meridian that also passes through the observer's zenith, as shown in Figure 5 (b).

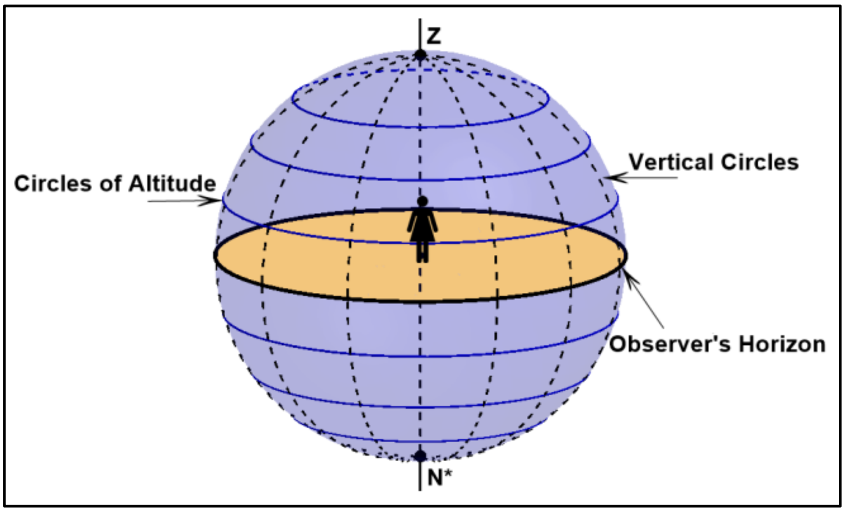
Figure 5 - (a) shows a meridian representation, which are the great circles whose planes contain the two celestial poles; on the other hand (b) shows the local meridian, which is the meridian that also passes through the observer's zenith



Source: Authors (2021)

Circles of the celestial sphere parallel to the horizon are called circles of altitude. A vertical circle is any great semicircle of the celestial sphere whose plane contains the vertical of the place. Vertical circles begin at the Zenith and end at the Nadir, as shown in Figure 6.

Figure 6 - Representation of altitude circles and vertical circles. As a reference, the figure shows the observer's horizon and the Zenith and Nadir points



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With the above definitions, we now introduce three coordinate systems commonly used to locate objects on the celestial sphere: horizontal, equatorial and hourly coordinate systems.

3.1 Horizontal Coordinates

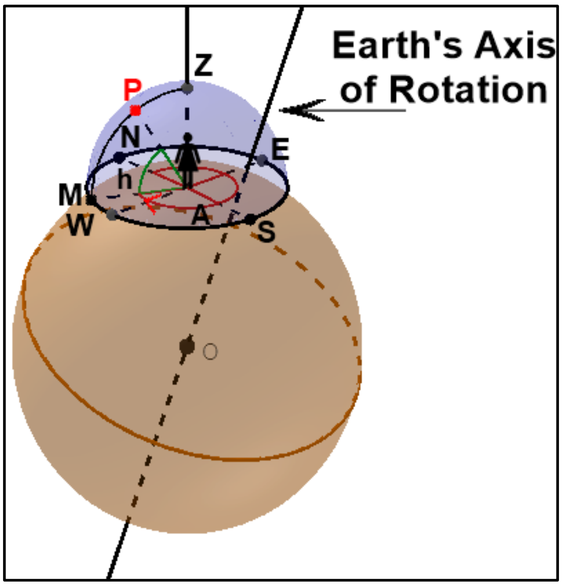
This system uses the observer's horizon plane as a reference. We will present below the coordinates of this system.

(Altitude) - Altitude is the arc measured over the vertical circle of the star P starting at the horizon and ending at the star. It is denoted by the letter h and measured between −90° and 90°.

In Figure 7, the altitude of a point P in the celestial sphere is represented by the arc $\widehat{MP}$.

$-90° \leq h \leq 90°$

Figure 7 - Illustration of the horizontal coordinate system of an observer's frame of reference. The Altitude and Azimuth coordinates are represented in the southern hemisphere. To aid interpretation, the figure illustrates the Earth in brown and the celestial sphere in blue separately



Source: Authors (2021)

(Zenith Distance) - The altitude complement is called the zenith distance, that is, the measure of the arc over the vertical circle starting at the Zenith and ending at the star. It is denoted by the letter z and measured between 0° and 180°.

Thereby:

$h + z = 90°$

and

$0° \leq z \leq 360°$

(Azimuth) - The arc measured over the horizon from the north point to the vertical circle of the star is called the azimuth. Denoted by the letter A, the azimuth varies between 0° and 360°, measured positively towards West (counterclockwise in the northern hemisphere, clockwise in the southern hemisphere).

In Figure 7, the azimuth A of a point P in the celestial sphere is represented by the arc $\widehat{SM}$.

$0° \leq A \leq 360°$

The horizontal system is a local system, as the coordinates of any star depend on the observer's location and on the local time.

3.2 Equatorial Coordinates

This system uses the celestial equator as a reference plane. Thus, their coordinates do not depend on the observer's position on the terrestrial globe. To present the coordinates of this system, we should first establish some concepts.

The circle in the celestial sphere formed by the apparent motion of the Sun in the frame of the Earth is called ecliptic. The ecliptic intersects with the celestial equator at two points: Aries point and Libra point. The Aries point is a point on the celestial equator occupied by the Sun at the March equinox, that is, when the Sun crosses the equator from the southern hemisphere and is denoted by γ. The Libra point is the point diametrically opposed to the Aries Point. It is the point occupied by the Sun at the September equinox, that is, when the Sun crosses the equator from the northern hemisphere and is denoted by Ω.

We can now introduce the coordinates of the equatorial system:

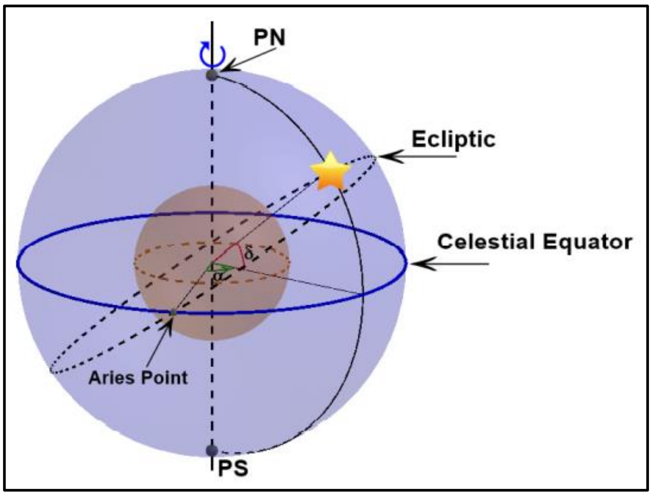
(Right ascension) - Right ascension is the arc measured over the celestial equator, starting at the meridian that passes through the Aries point and ending at the star's meridian. Denoted by α, it varies between 0° and 360°, measured eastward.

Check out Figure 8:

$0° \leq \alpha \leq 360°$

It is also common for the right ascension coordinate to be measured in hours: $0h \leq \alpha \leq 24h$.

Figure 8 - Illustration of the equatorial coordinate system of the Earth's reference frame. The Right Ascension and Declination coordinates are represented (with orientations). To aid interpretation, the figure illustrates the Earth in brown and the celestial sphere in blue separately



Source: Authors (2021)

(Declination) - Declination is the arc measured over the star's meridian, starting at the celestial equator and ending at the star. It is denoted by δ and measured between −90° and 90°.

See Figure 8:

$-90° \leq \delta \leq 90°$

(Polar Distance) - The complement of the declination is called the polar distance, that is, the arc over the meridian of the star, starting at the North Pole and ending at the star. It is denoted by Δ and measured between 0° and 180°.

Thereby:

$\delta + \Delta = 90°$

and

$0° \leq \Delta \leq 180°$

The coordinates of a star represented in the equatorial system do not depend on the place and time of observation.

3.3 Hourly Coordinates

The hourly coordinate system, like the equatorial system, has the plane of the equator as its fundamental plane. Below, we present the coordinates of this system.

(Declination) - The declination is the same as defined for equatorial coordinates.

Check out Figure 9, the arc $\widehat{PB}$ represents the declination of the star P:

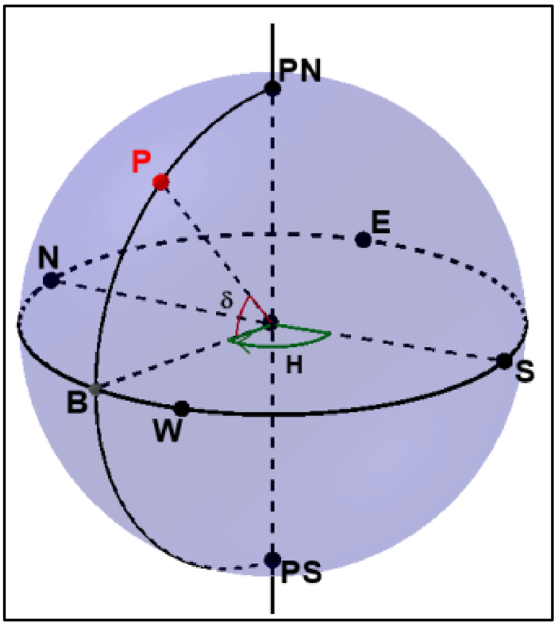
$$-90° \leq \delta \leq 90°$$

(Hour Angle) - Hour angle is the measure of the arc along the equator measured between the star's meridian and the local meridian. Denoted by H, it ranges from 0° to 360° counted positively towards West.

In Figure 9, the arc $\widehat{BS}$ represents the hour angle of the star P:

$$0° \leq H \leq 360°$$

Figure 9 - Illustration of the hourly coordinate system. The Hour Angle and Declination coordinates are represented (with orientations)



Source: Authors (2021)

It is also common for the hour angle to be measured in hours: $0h \leq H \leq 24h$.

The coordinates of a star represented in the hourly system are insensitive to place and time of observation, just as in the equatorial system.

4 COORDINATES IN THE TERRESTRIAL SPHERE - GEOGRAPHICAL COORDINATES

We now review the spherical system of geographic coordinates. In this system the Earth is assumed to present a perfecly spherical geometry. Great circles that pass through the geographical poles are called terrestrial meridians. This allows us to define the Geographic Longitude and Latitude as follows:

(Geographic Longitude) - The meridian that passes through Greenwich (city in south-east London, England) is called the Greenwich Meridian. Longitude is an arc measured along the geographic equator between the Greenwich and the local meridians. It is denoted by λ and measured in degrees, from zero to 180° to East or West, starting from the Greenwich Meridian.

In practice, a negative sign is used to indicate the W position and a positive sign for the E position. Thus, it is possible to write

$-180°(W) \leq \lambda \leq 180°(E)$

(Geographic Latitude) - Geographic latitude is the arc of the meridian, measured in degrees, from a point to the terrestrial equator. Latitude is denoted by Ø and is measured from zero to 90° to the north or south of the terrestrial equator.

In practice, a negative sign is used to indicate the S position and a positive sign for the N position. Thus, it is possible to write

$-90°(S) \leq \phi \leq 90°(N)$

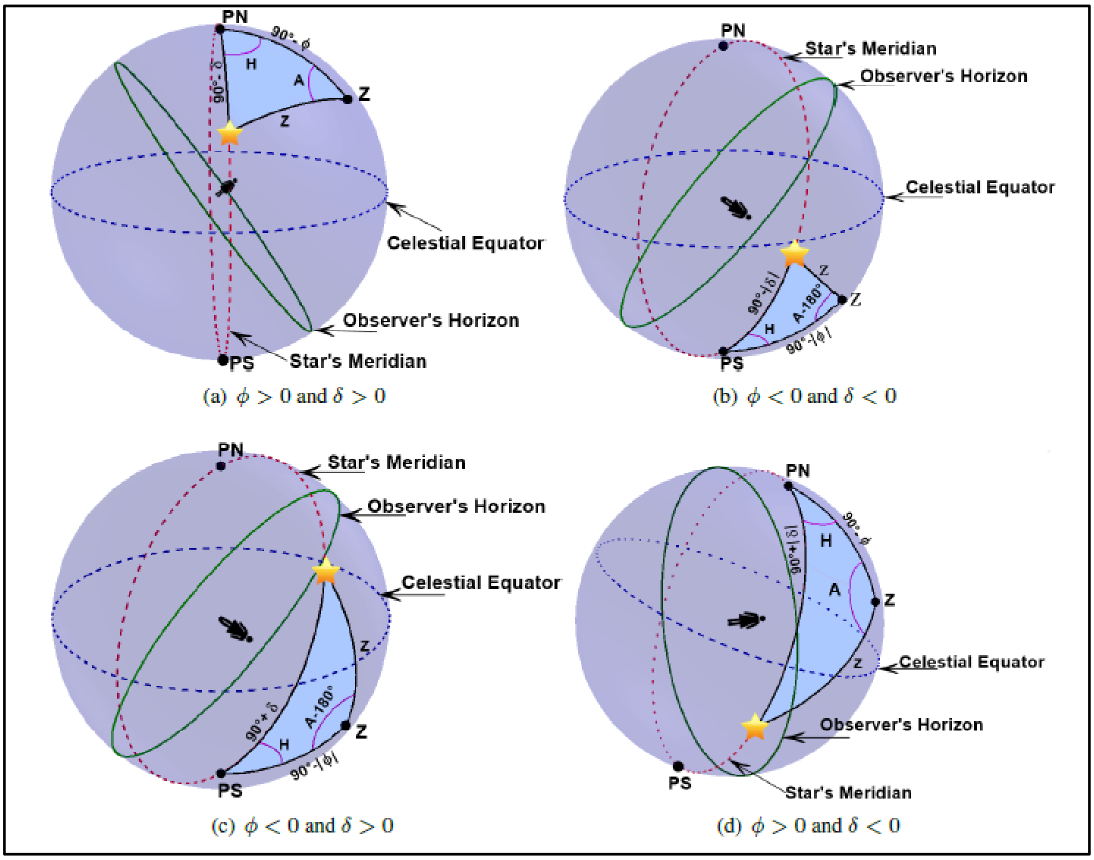
5 CONNECTIONS BETWEEN COORDINATES IN THE CELESTIAL SPHERE

In this section, we will establish some connections between the coordinates of the different systems presented previously (Zenith Distance, Hour Angle, Azimuth, Declination and Geographic Latitude). The deductions below are well known in astronomy. However, we present here all mathematical steps in detail, what is not common in the literature.

We will define, first of all, the concept of an astronomical position triangle. It is called astronomical position triangle of a certain star, the triangle located in the celestial sphere whose vertices are the star, the observer's zenith and the elevated pole (the pole located above the horizon).

It is necessary to note that, as the positions of the observer and the observed star vary, the ways of measuring the sides of the astronomical position triangle are altered. This occurs since the geographic latitude of the observer ϕ is positive when the observer is in the northern hemisphere and negative when he (or she) is in the southern hemisphere; likewise, the declination of the star δ is positive when the star is in the northern hemisphere and is negative when the star is in the southern hemisphere. As shown in Figure 10, there are four cases to consider:: "$\phi > 0$ and $\delta > 0$", "$\phi < 0$ and $\delta < 0$", "$\phi < 0$ and $\delta > 0$" and "$\phi > 0$ and $\delta < 0$".

Figure 10 - Examples of the four possibilities of astronomical position triangles varying ϕ and δ



(a) $\phi > 0$ and $\delta > 0$ (b) $\phi < 0$ and $\delta < 0$

(c) $\phi < 0$ and $\delta > 0$ (d) $\phi > 0$ and $\delta < 0$

Source: Authors (2021)

In the first case, when $\phi > 0$and $\delta > 0$, we have:

- The arc between the zenith and the pole measures $90° - \phi$;
- The arc between the pole and the star measures 90° − δ.

In the second case, when $\phi < 0$ and $\delta < 0$, we have:

- The arc between the zenith and the pole measures $90° - |\phi|$ (or $90° + \phi$);
- The arc between the pole and the star measures $90° - |\delta|$ (or 90° + δ).

In the third case, when $\phi < 0$ and $\delta > 0$, we have:

- The arc between the zenith and the pole measures $90^\circ - |\phi|$ (or $90^\circ + \phi$);
- The arc between the pole and the star measures $90^\circ + \delta$.

In the fourth case, when $\phi > 0$ and $\delta < 0$, we have:

- The arc between the zenith and the pole measures $90^\circ - \phi$;
- The arc between the pole and the star measures $90^\circ + |\delta|$ (or $90^\circ - \delta$).

Furthermore,

- The arc between the zenith and the star is the star's zenith distance z;
- The angle with vertex in the zenith measures the azimuth A of the star when in the northern hemisphere, when the zenith is in the southern hemisphere this angle measures A – 180°;
- The angle with vertex at the pole measures the hourly angle H of the star;
- The angle with a vertex in the star is called the paralactic angle.

Although there are four distinct cases, we have the following theorem:

Theorem 5. The zenith distance z, the hour angle H, the azimuth A, the declination δ and the geographic latitude ϕ are governed according to the following formulas, regardless of the hemisphere:

$$\cos(z) = \sin(\phi).\sin(\delta) + \cos(\phi).\cos(\delta).\cos(H) \quad (28)$$

$$\sin(\delta) = \sin(\phi).\cos(z) + \cos(\phi).\sin(z).\cos(A) \quad (29)$$

$$\sin(H).\cot(A) = \tan(\delta).\cos(\phi) - \sin(\phi).\cos(H) \quad (30)$$

$$\sin(z).\cos(A) = \cos(\phi).\sin(\delta) - \sin(\phi).\cos(\delta).\cos(H) \quad (31)$$

$$\cos(\delta).\cos(H) = \cos(\phi).\cos(z) - \sin(\phi).\sin(z).\cos(A) \quad (32)$$

In addition, when the observer is in the northern hemisphere:

$$\sin(z).\sin(A) = \sin(H).\cos(\delta) \quad (33)$$

$$\sin(A).\cot(H) = \cos(\phi).\cot(z) - \sin(\phi).\cos(A) \quad (34)$$

Furthermore, when the observer is in the southern hemisphere:

$$\sin(z).\sin(A) = -\sin(H).\cos(\delta) \tag{35}$$

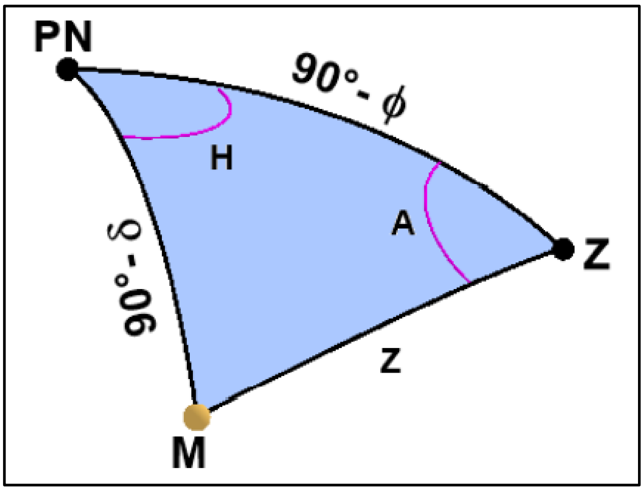
$$\sin(A).\cot(H) = \sin(\phi).\cos(A) - \cos(\phi).\cot(z) \tag{36}$$

Proof: Indeed,

First case: $\phi > 0 \, e \, \delta > 0$

Consider point M the star, point P the north celestial pole and point Z the zenith, as shown in Figure 11. It is known that z is the zenith distance, A is the azimuth and H is the hourly angle of the star.

Figure 11 - Astronomical position triangle where $\phi > 0 \; and \; \delta > 0$



Source: Authors (2021)

From Theorem 1, applied to the astronomical position triangle ZMP, it is observed:

$$\cos(z) = \cos(90° - \delta).\cos(90° - \phi) + \sin(90° - \delta).\sin(90° - \phi).\cos(H)$$

Using the expressions (40) and (42) from Appendix A, it follows that

$$\cos(z) = \sin(\phi).\sin(\delta) + \cos(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (28) for the first case.

Applying once again Theorem 1 to the ZMP astronomical position triangle, it is observed in an analogous way that:

$$\cos(90° - \delta) = \cos(90° - \phi).\cos(z) + \sin(90° - \phi).\sin(z).\cos(A)$$

Using the expressions (40) and (42) from Appendix A, it follows:

$$\sin(\delta) = \sin(\phi).\cos(z) + \cos(\phi).\sin(z).\cos(A)$$

Which proves the Equation (29) for the first case.

From Theorem 4:

$$\sin(H).\cot(A) = \sin(90^\circ - \phi).\cot(90^\circ - \delta) - \cos(90^\circ - \phi).\cos(H)$$

Using the expressions (40), (42) and (45) from Appendix A, it follows:

$$\sin(H).\cot(A) = \tan(\delta).\cos(\phi) - \sin(\phi).\cos(H)$$

Which proves the Equation (30) for the first case.

From Theorem 3, we have

$$\sin(z).\cos(A) = \sin(90^\circ - \phi).\cos(90^\circ - \delta) - \cos(90^\circ - \phi).\sin(90^\circ - \delta).\cos(H)$$

Using the expressions (40) and (42) from Appendix A, it follows:

$$\sin(z).\cos(A) = \cos(\phi).\sin(\delta) - \sin(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (31) for the first case.

Once again, from Theorem 3, it follows that

$$\sin(90^\circ - \delta).\cos(H) = \sin(90^\circ - \phi).\cos(z) - \cos(90^\circ - \phi).\sin(z).\cos(A)$$

Using the expressions (40) and (42) from Appendix A, it follows that

$$\cos(\delta).\cos(H) = \cos(\phi).\cos(z) - \sin(\phi).\sin(z).\cos(A)$$

Which proves the Equation (32) for the first case.

From Theorem 2, applied to the ZMP astronomical position triangle, follows:

$$\frac{\sin(z)}{\sin(H)} = \frac{\sin(90^\circ - \delta)}{\sin(A)}$$

Using the expression (40) from Appendix A, it follows:

$$\frac{\sin(z)}{\sin(H)} = \frac{\cos(\delta)}{\sin(A)}$$

In other words

$$\sin(z).\sin(A) = \sin(H).\cos(\delta)$$

Which proves the Equation (33) for the first case.

From Theorem 4:

$$\sin(A).\cot(H) = \sin(90° - \phi).\cot(z) - \cos(90° - \phi).\cos(A)$$

Using the expressions (40) and (42) from Appendix A, it follows that

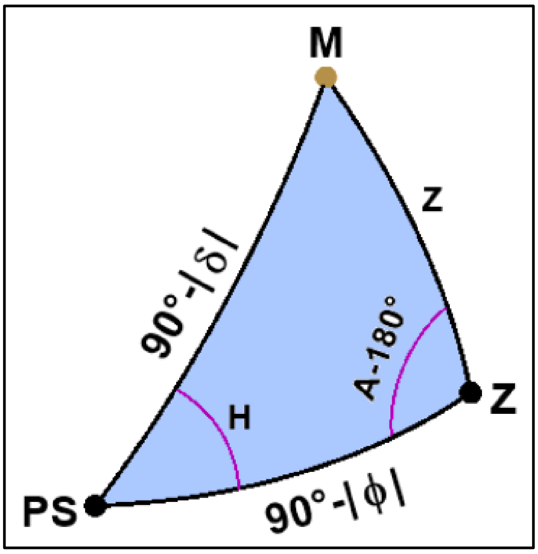
$$\sin(A).\cot(H) = \cos(\phi).\cot(z) - \sin(\phi).\cos(A)$$

Which proves the Equation (34) for the first case.

Second case: $\phi < 0$ e $\delta < 0$

Consider point M the star, point P the south celestial pole and point Z the zenith, as shown in Figure 12. It is known that z is the zenith distance, A is the azimuth and H is the hourly angle of the star

Figure 12 - Astronomical position triangle where ∅< 0 and δ < 0



Source: Authors (2021)

From Theorem 1, applied to the astronomical position triangle ZMP, it is observed:

$$\cos(z) = \cos(90° + \delta).\cos(90° + \phi) + \sin(90° + \delta).\sin(90° + \phi).\cos(H)$$

Using the expressions (43) and (41) from Appendix A, it follows that

$$\cos(z) = (-\sin(\delta)).(-\sin(\phi)) + \cos(\delta).\cos(\phi).\cos(H)$$

Implying

$$\cos(z) = \sin(\phi).\sin(\delta) + \cos(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (28) for the second case.

Applying once again Theorem 1 to the triangle of astronomical position ZMP, it is observed in an analogous way that

$$\cos(90^\circ + \delta) = \cos(90^\circ + \phi).\cos(z) + \sin(90^\circ + \phi).\sin(z).\cos(A - 180^\circ)$$

Using the expressions (39), (41), (43) and (47) from Appendix A, it follows that

$$-\sin(\delta) = -\sin(\phi).\cos(z) + \cos(\phi).\sin(z).(-\cos(A))$$

Implying

$$-\sin(\delta) = -\sin(\phi).\cos(z) - \cos(\phi).\sin(z).\cos(A)$$

In other words

$$\sin(\delta) = \sin(\phi).\cos(z) + \cos(\phi).\sin(z).\cos(A)$$

Which proves the Equation (29) for the second case.

From Theorem 4, we have

$$\sin(H).\cot(A - 180^\circ) = \sin(90^\circ + \phi).\cot(90^\circ + \delta) - \cos(90^\circ + \phi).\cos(H)$$

Using the expressions (41), (43), (44) and (46) from Appendix A, it follows that

$$-\sin(H).\cot(A) = -\tan(\delta).\cos(\phi) + \sin(\phi).\cos(H)$$

Implying

$$\sin(H).\cot(A) = \tan(\delta).\cos(\phi) - \sin(\phi).\cos(H)$$

Which proves the Equation (30) for the second case.

From Theorem 3, we have

$$\sin(z).\cos(A - 180^\circ) = \sin(90^\circ + \phi).\cos(90^\circ + \delta) - \cos(90^\circ + \phi).\sin(90^\circ + \delta).\cos(H)$$

Using the expressions (41), (43) and (47) from Appendix A, it follows:

$$-\sin(z).\cos(A) = -\cos(\phi).\sin(\delta) + \sin(\phi).\cos(\delta).\cos(H)$$

Implying

$$\sin(z).\cos(A) = \cos(\phi).\sin(\delta) - \sin(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (31) for the second case.

Over again, from Theorem 3, we have:

$$\sin(90^\circ + \delta).\cos(H) = \sin(90^\circ + \phi).\cos(z) - \cos(90^\circ + \phi).\sin(z).\cos(A - 180^\circ)$$

Using the expressions (41), (43) and (47) from Appendix A, it follows:

$$\cos(\delta).\cos(H) = \cos(\phi).\cos(z) - \sin(\phi).\sin(z).\cos(A)$$

Which proves the Equation (32) for the second case.

From Theorem 2, applied to the ZMP astronomical position triangle, it follows that

$$\frac{\sin(z)}{\sin(H)} = \frac{\sin(90^\circ + \delta)}{\sin(A - 180^\circ)}$$

Using the expressions (41) and (48) from Appendix A, it follows:

$$\frac{\sin(z)}{\sin(H)} = \frac{\cos(\delta)}{-\sin(A)}$$

In other words

$$\sin(z).\sin(A) = -\sin(H).\cos(\delta)$$

Which proves the Equation (35) for the second case.

Again, from Theorem 4:

$$\sin(A - 180^\circ).\cot(H) = \sin(90^\circ + \phi).\cot(z) - \cos(90^\circ + \phi).\cos(A - 180)$$

Using the expressions (41), (43), (47) and (48) from Appendix A, it follows that

$$-\sin(A).\cot(H) = \cos(\phi).\cot(z) - \sin(\phi).\cos(A)$$

Implying

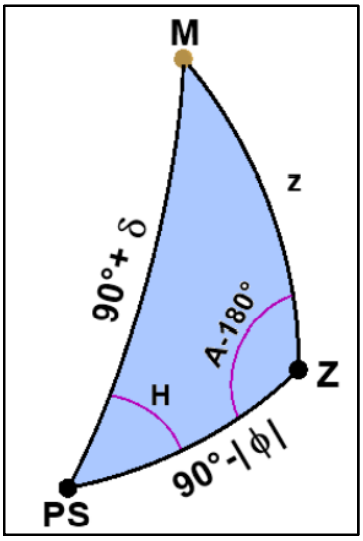
$$\sin(A).\cot(H) = \sin(\phi).\cos(A) - \cos(\phi).\cot(z)$$

Which proves the Equation (36) for the second case.

Third case: $\phi < 0 \text{ e } \delta > 0$

Consider point M the star, point P the south celestial pole and point Z the zenith, as shown in Figure 13. It is known that z is the zenith distance, A is the azimuth and H is the hourly angle of the star.

Figure 13 - Astronomical position triangle where $\phi < 0$ and $\delta > 0$



Source: Authors (2021)

From Theorem 1, applied to the astronomical position triangle ZMP, it is observed:

$$\cos(z) = \cos(90° + \delta).\cos(90° + \phi) + \sin(90° + \delta).\sin(90° + \phi).\cos(H)$$

Using the expressions (41) and (43) from Appendix A, it follows that

$$\cos(z) = (-\sin(\phi)).(-\sin(\delta)) + \cos(\phi).\cos(\delta).\cos(H)$$

In other words

$$\cos(z) = \sin(\phi).\sin(\delta) + \cos(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (28) for the third case.

Applying once again Theorem 1 to the ZMP astronomical position triangle, it is observed in an analogous way that:

$$\cos(90° + \delta) = \cos(90° + \phi).\cos(z) + \sin(90° + \phi).\sin(z).\cos(A - 180°)$$

Using the expressions (41), (43) and (47) from Appendix A, it follows:

$$-\sin(\delta) = -\sin(\phi).\cos(z) + \cos(\phi).\sin(z).(-\cos(A))$$

Implying

$$-\sin(\delta) = -\sin(\phi).\cos(z) - \cos(\phi).\sin(z).\cos(A)$$

In other words

$$\sin(\delta) = \sin(\phi).\cos(z) + \cos(\phi).\sin(z).\cos(A)$$

Which proves the Equation (29) for the third case.

From Theorem 4

$$\sin(H).\cot(A - 180^\circ) = \sin(90^\circ + \phi).\cot(90^\circ + \delta) - \cos(90^\circ + \phi).\cos(H)$$

Using the expressions (41), (43), (44) and (46) from Appendix A, it follows that

$$-\sin(H).\cot(A) = -\tan(\delta).\cos(\phi) + \sin(\phi).\cos(H)$$

Implying

$$\sin(H).\cot(A) = \tan(\delta).\cos(\phi) - \sin(\phi).\cos(H)$$

Which proves the Equation (30) for the third case.

From Theorem 3

$$\sin(z).\cos(A - 180^\circ) = \sin(90^\circ + \phi).\cos(90^\circ + \delta) - \cos(90^\circ + \phi).\sin(90^\circ + \delta).\cos(H)$$

Using the expressions (41), (43) and (47) from Appendix A, it follows that

$$-\sin(z).\cos(A) = -\cos(\phi).\sin(\delta) + \sin(\phi).\cos(\delta).\cos(H)$$

Implying

$$\sin(z).\cos(A) = \cos(\phi).\sin(\delta) - \sin(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (31) for the third case.

Again, from Theorem 3, we have:

$$\sin(90^\circ + \delta).\cos(H) = \sin(90^\circ + \phi).\cos(z) - \cos(90^\circ + \phi).\sin(z).\cos(A - 180^\circ)$$

Using the expressions (41), (43) and (47) from Appendix A, it follows that

$$\cos(\delta).\cos(H) = \cos(\phi).\cos(z) - \sin(\phi).\sin(z).\cos(A)$$

Which proves the Equation (32) for the third case.

From Theorem 2, applied to the ZMP astronomical position triangle, it follows that

$$\frac{\sin(z)}{\sin(H)} = \frac{\sin(90^\circ + \delta)}{\sin(A - 180^\circ)}$$

Using the expressions (41) and (48) from Appendix A, it follows that

$$\frac{\sin(z)}{\sin(H)} = \frac{\cos(\delta)}{-\sin(A)}$$

In other words

$$\sin(z).\sin(A) = -\sin(H).\cos(\delta)$$

Which proves the Equation (35) for the third case.

Once again, from Theorem 4, we have:

$$\sin(A-180^\circ).\cot(H) = \sin(90^\circ+\phi).\cot(z) - \cos(90^\circ+\phi).\cos(A-180)$$

Using the expressions (41), (43), (47) and (48) from Appendix A, it follows that

$$-\sin(A).\cot(H) = \cos(\phi).\cot(z) - \sin(\phi).\cos(A)$$

Implying

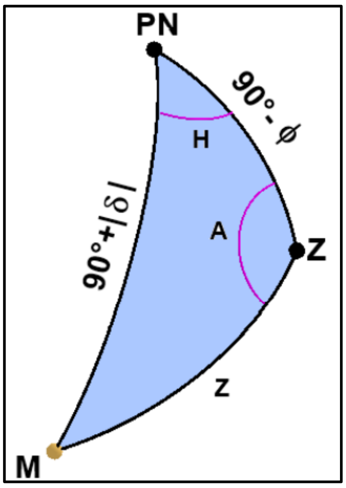
$$\sin(A).\cot(H) = \sin(\phi).\cos(A) - \cos(\phi).\cot(z)$$

Which proves the Equation (36) for the third case.

Fourth case $\phi > 0$ e $\delta < 0$

Consider point M the star, point P the north celestial pole and point Z the zenith, as shown in Figure 14. It is known that z is the zenith distance, A is the azimuth and H is the hourly angle of the star.

Figure 14 - Astronomical position triangle where $\phi > 0$ and $\delta < 0$



Source: Authors (2021)

From Theorem 1, applied to the ZMP astronomical position triangle, it is observed:

$$\cos(z) = \cos(90° - \delta).\cos(90° - \phi) + \sin(90° - \delta).\sin(90° - \phi).\cos(H)$$

Using the expressions (40) and (42) from Appendix A, it follows that

$$\cos(z) = \sin(\phi).\sin(\delta) + \cos(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (28) for the fourth case.

Once again applying Theorem 1 to the ZMP astronomical position triangle, it is observed that:

$$\cos(90° - \delta) = \cos(90° - \phi).\cos(z) + \sin(90° - \phi).\sin(z).\cos(A)$$

Using the expressions (40) and (42) from Appendix A, it follows that

$$\sin(\delta) = \sin(\phi).\cos(z) + \cos(\phi).\sin(z).\cos(A)$$

Which proves the Equation (29) for the fourth case.

From Theorem 4, we have

$$\sin(H).\cot(A) = \sin(90° - \phi).\cot(90° - \delta) - \cos(90° - \phi).\cos(H)$$

Using the expressions (40), (42) and (45) from Appendix A, it follows:

$$\sin(H).\cot(A) = \tan(\delta).\cos(\phi) - \sin(\phi).\cos(H)$$

Which proves the Equation (30) for the fourth case.

From Theorem 3, we have:

$$\sin(z).\cos(A) = \sin(90° - \phi).\cos(90° - \delta) - \cos(90° - \phi).\sin(90° - \delta).\cos(H)$$

Using the expressions (40) and (42) from Appendix A, it follows that

$$\sin(z).\cos(A) = \cos(\phi).\sin(\delta) - \sin(\phi).\cos(\delta).\cos(H)$$

Which proves the Equation (31) for the fourth case.

Once again, from Theorem 3, it follows:

$$\sin(90° - \delta).\cos(H) = \sin(90° - \phi).\cos(z) - \cos(90° - \phi).\sin(z).\cos(A)$$

Using the expressions (40) and (42) from Appendix A, it follows:

$$\cos(\delta).\cos(H) = \cos(\phi).\cos(z) - \sin(\phi).\sin(z).\cos(A)$$

Which proves the Equation (32) for the fourth case.

From Theorem 2, applied to the ZMP astronomical position triangle, it follows that

$$\frac{\sin(z)}{\sin(H)} = \frac{\sin(90° - \delta)}{\sin(A)}$$

Using the expression (40) from Appendix A, it follows:

$$\frac{\sin(z)}{\sin(H)} = \frac{\cos(\delta)}{\sin(A)}$$

In other words

$$\sin(z).\sin(A) = \sin(H).\cos(\delta)$$

Which proves the Equation (33) for the fourth case.

Again, from Theorem 4:

$$\sin(A).\cot(H) = \sin(90° - \phi).\cot(z) - \cos(90° - \phi).\cos(A)$$

Using the expressions (40) and (42) from Appendix A, it follows:

$$\sin(A).\cot(H) = \cos(\phi).\cot(z) - \sin(\phi).\cos(A)$$

Which proves the Equation (34) for the fourth case.

6 FINAL REMARKS

The wide applicability of spherical geometry and trigonometry in astronomy, geodesy, navigation and other areas, justifies the importance of studying this subject. Despite being a theme studied for many decades, there is always something to add or approach differently.

Concerning Positional Astronomy, in this article some formulas have been presented that exhibit connections between coordinates in the celestial sphere. Seven formulas were presented connecting the coordinates Zenith Distance, Hour Angle, Azimuth, Declination and Geographic Latitude. Five of these formulas are independent of the observer's position and two of them vary slightly according to the hemisphere in which the observer is located. Such formulas are important because they allow the analytical study of the apparent motion of stars in the celestial sphere. In addition to displaying these formulas, which are not common in Positional Astronomy texts, we did the detailed proofs for each one. In order to prove them, we consider four cases, depending on the geographic latitude ϕ and the declination of the star of interest δ. In this way, we make one more small contribution for those who want to learn this beautiful geometry.

APPENDIX A - WELL-KNOWN TRIGONOMETRIC IDENTITIES FOR PLANE GEOMETRY

This Appendix presents some well-known trigonometric identities for plane geometry. Such equalities will be constantly requested throughout the text of this study

$$\cos(a \pm b) = \cos(a).\cos(b) \mp \sin(a).\sin(b) \tag{37}$$

$$\sin(a \pm b) = \sin(a).\cos(b) \pm \cos(a).\sin(b) \tag{38}$$

$$\cos(-a) = \cos(a) \tag{39}$$

$$\sin(90° - a) = \cos(a) \tag{40}$$

$$\sin(90° + a) = \cos(a) \tag{41}$$

$$\sin(90° + a) = \cos(a) \tag{42}$$

$$\cos(90° + a) = -\sin(a) \tag{43}$$

$$\cot(90° + a) = -\tan(a) \tag{44}$$

$$\cot(90° - a) = \tan(a) \tag{45}$$

$$\cot(a - 180°) = -\cot(a) \tag{46}$$

$$\cos(a - 180°) = -\cos(a) \tag{47}$$

$$\sin(a - 180°) = \sin(a) \tag{48}$$

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