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A Variational Formulation for the Relativistic Klein-Gordon Equation

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Abstract

This article develops a variational formulation for the relativistic Klein-Gordon equation. The main results are obtained through a connection between classical and quantum mechanics. Such a connection is established through the definition of normal field and its relation with the wave function concept.

Keywords: *Quantum mechanics; Wave function; Normal field*

1 The Newtonian approach

About the references, this work is based on the book "A Classical Description of Variational Quantum Mechanics and Related Models" [5], published by Nova Science Publishers. Details on the Sobolev Spaces involved may be found in [1, 4]. For standard references in quantum mechanics, we refer to [3, 6, 7] and the non-standard [2].

Finally, we emphasize this article is not about Bohmian mechanics, even though the David Bohm work has been always inspiring.

In this section, specifically for a free particle context, we shall obtain a close relationship between classical and quantum mechanics.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$, on which we define a position field, in a free volume context, denoted by $\mathbf{r} : \Omega \times [0,T] \to \mathbb{R}^3$, where $[0,T]$ is a time interval.

Suppose also an associated density distribution scalar field is given by $(\rho \circ \mathbf{r}) : \Omega \times [0,T] \to [0,+\infty)$, so that the kinetics energy for such a system, denoted by $J : U \times V \to \mathbb{R}$, is defined as

$$J(\mathbf{r},\rho) = \frac{1}{2}\int_0^T \int_\Omega \rho(\mathbf{r}(\mathbf{x},t))\frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t} \cdot \frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t}\sqrt{g}\, d\mathbf{x}dt,$$

subject to

$$\int_\Omega \rho(\mathbf{r}(\mathbf{x},t))\sqrt{g}\, d\mathbf{x} = m, \text{ on } [0,T],$$

where m is the total system mass, t denotes time and $d\mathbf{x} = dx_1\, dx_2\, dx_3$.

Here,

$$\begin{aligned} U &= \{\mathbf{r} \in W^{1,2}(\Omega \times [0,T]) \; : \; \mathbf{r}(\mathbf{x},0) = \mathbf{r}_0(\mathbf{x}) \\ &\quad \text{and } \mathbf{r}(\mathbf{x},T) = \mathbf{r}_1(\mathbf{x}), \text{ in } \Omega\}, \end{aligned} \tag{1}$$

and

$$V = \{\rho(\mathbf{r}) \in L^2([0,T]; W^{1,2}(\Omega)) \; : \; \mathbf{r} \in U\}.$$

Also

$$\mathbf{g}_k = \frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial x_k},$$

$$g_{jk} = \mathbf{g}_j \cdot \mathbf{g}_k,$$

and

$$g = \det\{g_{jk}\}.$$

For such a standard Newtonian formulation, the kinetics energy takes into account just the tangential field given by the time derivative

$$\frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t}.$$

At this point, the idea is to complement such an energy with a new term which would consider also the variation of a normal field $\mathbf{n}$ and concerning distribution of curvature, such that

$$\mathbf{n} \cdot \frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t} = 0, \text{ in } \Omega \times [0,T].$$

So, with such statements in mind, we redefine the concerning energy, denoting it again by $J : U \times V \times V_1 \to \mathbb{R}$, as

$$\begin{aligned} J(\mathbf{r},\mathbf{n},\rho) &= -\frac{1}{2}\int_0^T \int_\Omega \rho(\mathbf{r}(\mathbf{x},t))\frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t}\cdot\frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t}\sqrt{g}\;d\mathbf{x}dt \\ &+\frac{\gamma}{2}\int_0^T\int_\Omega \hat{R}\sqrt{g}\;d\mathbf{x}dt, \end{aligned} \tag{2}$$

where $\gamma > 0$ is an appropriate constant,

$$\mathbf{g}_k = \frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial x_k},$$

$$g = det\{g_{ij}\},$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

$$\hat{R} = g^{ij}\hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i\; b_{jk},$$

$$b_{ij} = -\frac{1}{\sqrt{m}}\frac{\partial\left(\sqrt{\rho(\mathbf{r})}\mathbf{n}(\mathbf{r})\right)}{\partial x_j}\cdot \mathbf{g}_i,$$

$$b^i_j = g^{il}b_{lj},$$

and,

$$\{g^{ij}\} = \{g_{ij}\}^{-1},$$

$\forall i,j,k,l \in \{1,2,3\}$.

subject to

$$\mathbf{n}(\mathbf{r})\cdot\mathbf{n}(\mathbf{r}) = 1, \text{ in } \Omega\times[0,T],$$

$$\mathbf{n}(\mathbf{r})\cdot\frac{\partial \mathbf{r}}{\partial t} = 0, \text{ in } \Omega\times[0,T],$$

and

$$\int_\Omega \rho(\mathbf{r}(\mathbf{x},t))\sqrt{g}\;d\mathbf{x} = m, \text{ on } [0,T].$$

Here

$$V_1 = \{\mathbf{n}(\mathbf{r}) \in L^2(\Omega\times[0,T]) \;:\; \mathbf{r}\in U\}.$$

Thus, defining ϕ such that

$$|\phi| = \sqrt{\frac{\rho}{m}}$$

and already including the Lagrange multipliers concerning the restrictions, the final expression for the energy, denoted by $J : U \times V \times V_1 \times V_2 \times [V_3]^2 \to \mathbb{R}$, would be given by

$$\begin{aligned} J(\mathbf{r},\mathbf{n},\phi,E,\lambda_1,\lambda_2) &= -\frac{1}{2}\int_0^T\int_\Omega m|\phi(\mathbf{r}(\mathbf{x},t))|^2\frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t}\cdot\frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t}\sqrt{g}\;d\mathbf{x}dt \\ &+\frac{\gamma}{2}\int_0^T\int_\Omega \hat{R}\sqrt{g}\;d\mathbf{x}dt \\ &-m\int_0^T E(t)\left(\int_\Omega |\phi(\mathbf{r})|^2\;\sqrt{g}\;d\mathbf{x}-1\right)\;dt \\ &+\langle \lambda_1, \mathbf{n}\cdot\mathbf{n}-1\rangle_{L^2} \\ &+\left\langle \lambda_2, \mathbf{n}\cdot\frac{\partial \mathbf{r}}{\partial t}\right\rangle_{L^2}, \end{aligned} \tag{3}$$

where,

$$\begin{aligned} U \quad = \quad & \{\mathbf{r} \in W^{1,2}(\Omega \times [0,T]) \ : \ \mathbf{r}(\mathbf{x},0) = \mathbf{r}_0(\mathbf{x}) \\ & \text{and } \mathbf{r}(\mathbf{x},T) = \mathbf{r}_1(\mathbf{x}), \ \text{in } \Omega\}, \end{aligned} \tag{4}$$

$$V = \{\phi(\mathbf{r}) \in L^2([0,T]; W^{1,2}(\Omega;\mathbb{C})) \ : \ \mathbf{r} \in U\},$$

$$V_1 = \{\mathbf{n}(\mathbf{r}) \in L^2(\Omega \times [0,T]) \ : \ \mathbf{r} \in U\},$$

$$V_2 = L^2([0,T]),$$

$$V_3 = L^2(\Omega \times [0,T]),$$

and generically

$$\langle f, h \rangle_{L^2} = \int_0^T \int_\Omega f h \sqrt{g} \, d\mathbf{x} \, dt, \forall f, h \in L^2(\Omega \times [0,T]).$$

Moreover,

$$\mathbf{g}_k = \frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial x_k},$$

$$g = det\{g_{ij}\},$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

$$\hat{R} = g^{ij} \hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i \, b^*_{jk},$$

$$b_{ij} = -\frac{\partial \left(\phi(\mathbf{r})\mathbf{n}(\mathbf{r})\right)}{\partial x_j} \cdot \mathbf{g}_i,$$

$$b^i_j = g^{il} b_{lj},$$

and,

$$\{g^{ij}\} = \{g_{ij}\}^{-1},$$

$\forall i, j, k, l \in \{1, 2, 3\}$.

Finally, in particular for the special case in which

$$\mathbf{r}(\mathbf{x},t) \approx \mathbf{x},$$

so that

$$\frac{\partial \mathbf{r}(\mathbf{x},t)}{\partial t} \approx 0,$$

and

$$\mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial t} \approx 0,$$

we may set

$$\mathbf{n} = \mathbf{c},$$

where $\mathbf{c} \in \mathbb{R}^3$ is a constant such that

$$\mathbf{c} \cdot \mathbf{c} = 1,$$

and obtain

$$\mathbf{g}_k \approx \mathbf{e}_k,$$

where

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

is the canonical basis of $\mathbb{R}^3$.

Therefore, in such a case,

$$\frac{\gamma}{2}\int_0^T\int_\Omega \hat{R}\,\sqrt{g}\;d\mathbf{x}dt \approx \frac{\gamma T}{2}\sum_{k=1}^{3}\int_\Omega \frac{\partial\phi}{\partial x_k}\frac{\partial\phi^*}{\partial x_k}\;d\mathbf{x}.$$

Hence, we would also obtain

$$\begin{aligned} J(\mathbf{r},\mathbf{n},\phi,E,\lambda_1,\lambda_2)/T &\approx \tilde{J}(\phi,E) \\ &= \frac{\gamma}{2}\sum_{k=1}^{3}\int_\Omega \frac{\partial\phi}{\partial x_k}\frac{\partial\phi^*}{\partial x_k}\;d\mathbf{x} \\ &\quad -E\left(\int_\Omega |\phi|^2 d\mathbf{x} - 1\right). \end{aligned} \tag{5}$$

This last energy is just the standard Schrödinger one in a free particle context.

2 A brief note on the realistic context, the Klein-Gordon equation

Denoting by c the speed of light and

$$d\tilde{t}^2 = c^2dt^2 - dX_1^2 - dX_2^2 - dX_3^2,$$

in a relativistic free particle context, the Hilbert variational formulation could be extended, for a motion in a pseudo Riemannian relativistic C^1 class manifold M, where locally

$$M = \{\mathbf{r}(\mathbf{u})\;:\;\mathbf{u}\in\Omega\},$$

$$\mathbf{u} = (u_1,u_2,u_3,u_4)\in\mathbb{R}^4,$$

and

$$\mathbf{r}:\Omega\subset\mathbb{R}^4\to\mathbb{R}^4$$

point-wise stands for,

$$\mathbf{r}(\mathbf{u}) = (ct(\mathbf{u}),X_1(\mathbf{u}),X_2(\mathbf{u}),X_3(\mathbf{u})),$$

to a functional J_1 where denoting $\rho(\mathbf{r}) = |R(\mathbf{r})|^2$, the mass differential is given by

$$dm = \frac{\rho(\mathbf{r})}{\sqrt{1-v^2/c^2}}\sqrt{|g|}\;d\mathbf{u} = \frac{|R(\mathbf{r})|^2}{\sqrt{1-v^2/c^2}}\sqrt{|g|}\;d\mathbf{u},$$

the semi-classical kinetics energy differential is given by

$$\begin{aligned} dE_c &= \frac{\partial\mathbf{r}(\mathbf{u})}{\partial t}\cdot\frac{\partial\mathbf{r}(\mathbf{u})}{\partial t}\;dm \\ &= -\left(\frac{d\tilde{t}}{dt}\right)^2\;dm \\ &= -(c^2-v^2)\;dm, \end{aligned} \tag{6}$$

so that

$$dE_c = -c^2(\sqrt{1-v^2/c^2})|R(\mathbf{r})|^2\sqrt{|g|}\;d\mathbf{u},$$

and

$$\begin{aligned} J_1(\mathbf{r}, R, \mathbf{n}) &= -\int_\Omega dE_c + \frac{\gamma}{2}\int_\Omega \hat{R}\sqrt{|g|}\,d\mathbf{u} \\ &= c^2\int_\Omega |R(\mathbf{r})|^2\sqrt{1-v^2/c^2}\,\sqrt{|g|}\,d\mathbf{u} \\ &\quad +\frac{\gamma}{2}\int_\Omega \hat{R}\sqrt{|g|}\,d\mathbf{u}, \end{aligned} \tag{7}$$

subject to

$$\int_\Omega |R(\mathbf{r})|^2\,\sqrt{|g|}\,d\mathbf{u} = m,$$

where m is the particle mass at rest.

Moreover,

$$\mathbf{n}(\mathbf{r})\cdot\frac{\partial \mathbf{r}}{\partial \bar{t}} = 0, \text{ in } \Omega,$$

where

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \bar{t}} &= \frac{\partial \mathbf{r}}{\partial t}\frac{\partial t}{\partial \bar{t}} \\ &= \frac{\frac{\partial \mathbf{r}}{\partial t}}{\frac{\partial \bar{t}}{\partial t}} \\ &= \frac{\partial \mathbf{r}}{c\partial t}\frac{1}{\sqrt{1-v^2/c^2}}, \end{aligned} \tag{8}$$

and

$$\mathbf{n}(\mathbf{r})\cdot\mathbf{n}(\mathbf{r}) = 1, \text{ in } \Omega.$$

Where γ is an appropriate positive constant to be specified.

Also,

$$\mathbf{g}_k = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_k},$$

$$g = det\{g_{ij}\},$$

$$g_{ij} = \mathbf{g}_i\cdot\mathbf{g}_j,$$

where here, in this subsection, such a product is given by

$$\mathbf{y}\cdot\mathbf{z} = -y_0z_0 + \sum_{i=1}^{3} y_iz_i, \ \forall \mathbf{y} = (y_0, y_1, y_2, y_3), \ \mathbf{z} = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4,$$

$$\hat{R} = g^{ij}\hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i\, b^*_{jk},$$

$$b_{ij} = -\frac{1}{\sqrt{m}}\frac{\partial\left(R(\mathbf{r})\mathbf{n}(\mathbf{r})\right)}{\partial u_j}\cdot\mathbf{g}_i,$$

$$b^i_j = g^{il}b_{lj},$$

and,

$$\{g^{ij}\} = \{g_{ij}\}^{-1},$$

$\forall i, j, k, l \in \{1, 2, 3, 4\}$.

Finally,

$$v = \sqrt{\left(\frac{\partial X_1}{\partial t}\right)^2 + \left(\frac{\partial X_2}{\partial t}\right)^2 + \left(\frac{\partial X_3}{\partial t}\right)^2},$$

where,

$$\begin{aligned} \frac{\partial X_k(\mathbf{u})}{\partial t} &= \frac{\partial X_k(\mathbf{u})}{\partial u_j}\frac{\partial u_j}{\partial t} \\ &= \sum_{j=1}^{4} \frac{\frac{\partial X_k(\mathbf{u})}{\partial u_j}}{\frac{\partial t(\mathbf{u})}{\partial u_j}}, \ \forall k \in \{1, 2, 3\}. \end{aligned} \tag{9}$$

Here the Einstein sum convention holds.

Remark 2.1. *The role of the variable* $\mathbf{u}$ *concerns the idea of establishing a relation between* t, X_1, X_2 *and* X_3*. The dimension of* M *may vary with the problem in question.*

2.1 Obtaining the Klein-Gordon equation

Of particular interest is the case in which

$$\mathbf{u} = (t, x_1, x_2, x_3) = (t, \mathbf{x}) \in \mathbb{R}^4,$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

In such a case we could have, point-wise,

$$\mathbf{r}(\mathbf{x}, t) = (ct, X_1(t, \mathbf{x}), X_2(t, \mathbf{x}), X_3(t, \mathbf{x})),$$

and

$$M = \{\mathbf{r}(\mathbf{x}, t) \ : \ (\mathbf{x}, t) \in \Omega \times [0, T]\},$$

for an appropriate $\Omega \subset \mathbb{R}^3$.

Also, denoting $d\mathbf{x} = dx_1 dx_2 dx_3$, the mass differential would be given by

$$dm = \frac{\rho(\mathbf{r})}{\sqrt{1 - v^2/c^2}}\sqrt{-g}\, d\mathbf{x} = \frac{|R(\mathbf{r})|^2}{\sqrt{1 - v^2/c^2}}\sqrt{-g}\, d\mathbf{x},$$

the semi-classical kinetics energy differential would be expressed by

$$\begin{aligned} dE_c &= \frac{\partial \mathbf{r}(t, \mathbf{x})}{\partial t} \cdot \frac{\partial \mathbf{r}(t, \mathbf{x})}{\partial t}\, dm \\ &= -\left(\frac{d\bar{t}}{dt}\right)^2 dm \\ &= -(c^2 - v^2)\, dm, \end{aligned} \tag{10}$$

so that

$$dE_c = -c^2(\sqrt{1 - v^2/c^2})|R(\mathbf{r})|^2\sqrt{-g}\, d\mathbf{x},$$

where

$$d\bar{t}^2 = c^2 dt^2 - dX_1(t, \mathbf{x})^2 - dX_2(t, \mathbf{x})^2 - dX_3(t, \mathbf{x})^2,$$

and

$$\begin{aligned} J_1(\mathbf{r}, R, \mathbf{n}) &= -\int_0^T \int_\Omega dE_c \, dt + \frac{\gamma}{2} \int_0^T \int_\Omega \hat{R}\sqrt{-g} \, d\mathbf{x} \, dt \\ &= c^2 \int_0^T \int_\Omega |R(\mathbf{r})|^2 \sqrt{1 - v^2/c^2} \sqrt{-g} \, d\mathbf{x} \, dt \\ &+ \frac{\gamma}{2} \int_0^T \int_\Omega \hat{R}\sqrt{-g} \, d\mathbf{x} \, dt, \end{aligned} \tag{11}$$

subject to

$$R(\mathbf{r}(\mathbf{x}, 0)) = R_0(\mathbf{x})$$
$$R(\mathbf{r}(\mathbf{x}, T)) = R_1(\mathbf{x})$$

and

$$R(\mathbf{r}(\mathbf{x}, t)) = 0, \text{ on } \partial\Omega \times [0, T],$$

$$\int_\Omega |R(\mathbf{r})|^2 \sqrt{-g} \, d\mathbf{x} = m, \text{ on } [0, T],$$

$$\mathbf{n}(\mathbf{r}) \cdot \frac{\partial \mathbf{r}}{\partial \bar{t}} = 0, \text{ in } \Omega \times [0, T],$$

where

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \bar{t}} &= \frac{\partial \mathbf{r}}{\partial t} \frac{\partial t}{\partial \bar{t}} \\ &= \frac{\frac{\partial \mathbf{r}}{\partial t}}{\frac{\partial \bar{t}}{\partial t}} \\ &= \frac{\partial \mathbf{r}}{c \partial t} \frac{1}{\sqrt{1 - v^2/c^2}}, \end{aligned} \tag{12}$$

and

$$\mathbf{n}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) = 1, \text{ in } \Omega \times [0, T].$$

Also, we have denoted

$$x_0 = ct,$$
$$(x_0, \mathbf{x}) = (x_0, x_1, x_2, x_3),$$
$$\mathbf{g}_k = \frac{\partial \mathbf{r}(t, \mathbf{x})}{\partial x_k},$$
$$g = det\{g_{ij}\},$$
$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

where here again, such a product is given by

$$\mathbf{y} \cdot \mathbf{z} = -y_0 z_0 + \sum_{i=1}^{3} y_i z_i, \; \forall \mathbf{y} = (y_0, y_1, y_2, y_3), \; \mathbf{z} = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4,$$

$$\hat{R} = g^{ij}\hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i\, b^*_{jk},$$

$$b_{ij} = -\frac{1}{\sqrt{m}}\frac{\partial\left(R(\mathbf{r})\mathbf{n}(\mathbf{r})\right)}{\partial x_j}\cdot \mathbf{g}_i,$$

$$b^i_j = g^{il}b_{lj},$$

and,

$$\{g^{ij}\} = \{g_{ij}\}^{-1},$$

$\forall i, j, k, l \in \{0, 1, 2, 3\}$.

Finally, we would also have

$$v = \sqrt{\left(\frac{\partial X_1}{\partial t}\right)^2 + \left(\frac{\partial X_2}{\partial t}\right)^2 + \left(\frac{\partial X_3}{\partial t}\right)^2}.$$

In particular for the special case in which

$$\mathbf{r}(\mathbf{x}, t) \approx (ct, \mathbf{x}),$$

so that

$$\frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} \approx (c, 0, 0, 0),$$

and

$$\mathbf{n}\cdot\frac{\partial \mathbf{r}}{\partial t} \approx 0,$$

where we have set

$$\mathbf{n} = \mathbf{c} = (0, c_1, c_2, c_3).$$

Here $\mathbf{c} \in \mathbb{R}^4$ is a constant such that

$$\mathbf{c}\cdot\mathbf{c} = 1,$$

and thus we would obtain

$$\mathbf{g}_0 \approx (1, 0, 0, 0),\ \mathbf{g}_1 \approx (0, 1, 0, 0),\ \mathbf{g}_2 \approx (0, 0, 1, 0) \text{ and } \mathbf{g}_3 \approx (0, 0, 0, 1) \in \mathbb{R}^4.$$

Therefore, defining $\phi \in W^{1,2}(\Omega \times [0, T]; \mathbb{C})$ as

$$\phi(\mathbf{x}, t) = \frac{R(ct, \mathbf{x})}{\sqrt{m}},$$

we have

$$\begin{aligned}\frac{\gamma}{2}\int_0^T\int_\Omega \hat{R}\,\sqrt{g}\;d\mathbf{x}dt \quad &\approx \quad \frac{\gamma}{2}\int_0^T\int_\Omega\left(-\frac{1}{c^2}\frac{\partial\phi(\mathbf{x},t)}{\partial t}\frac{\partial\phi^*(\mathbf{x},t)}{\partial t}\right.\\ &\left.+\sum_{k=1}^3\frac{\partial\phi(\mathbf{x},t)}{\partial x_k}\frac{\partial\phi^*(\mathbf{x},t)}{\partial x_k}\right)\;d\mathbf{x}dt, \qquad (13)\end{aligned}$$

and

$$c^2 \int_0^T \int_\Omega |R(\mathbf{r})|^2 \sqrt{1-v^2/c^2}\,\sqrt{-g}\; d\mathbf{x}\; dt \approx mc^2 \int_0^T \int_\Omega |\phi(\mathbf{x},t)|^2 \; d\mathbf{x}dt.$$

Hence, we would also obtain

$$\begin{aligned} J(\mathbf{r},\mathbf{n},\phi,E,\lambda_1,\lambda_2) \quad \approx \quad & \frac{\gamma}{2}\left(\int_0^T \int_\Omega -\frac{1}{c^2}\frac{\partial \phi(\mathbf{x},t)}{\partial t}\frac{\partial \phi^*(\mathbf{x},t)}{\partial t}\; d\mathbf{x}dt \right. \\ & \left. + \sum_{k=1}^3 \int_\Omega \int_0^T \frac{\partial \phi(\mathbf{x},t)}{\partial x_k}\frac{\partial \phi^*(\mathbf{x},t)}{\partial x_k}\; d\mathbf{x}dt\right) \\ & + mc^2 \int_0^T \int_\Omega |\phi(\mathbf{x},t)|^2 \; d\mathbf{x}dt \\ & - m\int_0^T E(t)\left(\int_\Omega |\phi(\mathbf{x},t)|^2 d\mathbf{x} - 1\right)\; dt. \end{aligned} \tag{14}$$

The Euler Lagrange equations for such an energy are given by

$$\begin{aligned} & \frac{\gamma}{2}\left(\frac{1}{c^2}\frac{\partial^2 \phi(\mathbf{x},t)}{\partial t^2} - \sum_{k=1}^3 \frac{\partial^2 \phi(\mathbf{x},t)}{\partial x_k^2}\right) \\ & + mc^2 \phi(\mathbf{x},t) - E_1(t)\phi(\mathbf{x},t) = 0, \text{ in } \Omega, \end{aligned} \tag{15}$$

where,

$$\begin{aligned} \phi(\mathbf{x},0) &= \phi_0(\mathbf{x}), \text{ in } \Omega, \\ \phi(\mathbf{x},T) &= \phi_1(\mathbf{x}), \text{ in } \Omega, \\ \phi(\mathbf{x},t) &= 0, \text{ on } \partial\Omega \times [0,T] \end{aligned}$$

and $E_1(t) = mE(t)$.

Equation (15) is the relativistic Klein-Gordon one.

For $E_1(t) = E_1 \in \mathbb{R}$ (not time dependent), at this point we suggest a solution (and implicitly related time boundary conditions) $\phi(\mathbf{x},t) = e^{-\frac{iE_1 t}{\hbar}}\phi_2(\mathbf{x})$, where

$$\phi_2(\mathbf{x}) = 0, \text{ on } \partial\Omega.$$

Therefore, replacing this solution into equation (15), we would obtain

$$\left(\frac{\gamma}{2}\left(-\frac{E_1^2}{c^2\hbar^2}\phi_2(\mathbf{x}) - \sum_{k=1}^3 \frac{\partial^2 \phi_2(\mathbf{x})}{\partial x_k^2}\right) + mc^2\phi_2(\mathbf{x}) - E_1\phi_2(\mathbf{x})\right) e^{-\frac{iE_1 t}{\hbar}} = 0,$$

in Ω.

Denoting

$$E_2 = -\frac{\gamma E_1^2}{2c^2\hbar^2} + mc^2 - E_1,$$

the final eigenvalue problem would stand for

$$-\frac{\gamma}{2}\sum_{k=1}^3 \frac{\partial^2 \phi_2(\mathbf{x})}{\partial x_k^2} + E_2\phi_2(\mathbf{x}) = 0, \text{ in } \Omega$$

where E_1 is such that

$$\int_\Omega |\phi_2(\mathbf{x})|^2 \, d\mathbf{x} = 1.$$

Moreover, from (15), such a solution $\phi(\mathbf{x},t) = e^{-\frac{iE_1 t}{\hbar}}\phi_2(\mathbf{x})$ is also such that

$$\frac{\gamma}{2}\left(\frac{1}{c^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} - \sum_{k=1}^{3}\frac{\partial^2\phi(\mathbf{x},t)}{\partial x_k^2}\right)$$
$$+mc^2\phi(\mathbf{x},t) = i\hbar\frac{\partial\phi(\mathbf{x},t)}{\partial t}, \text{ in } \Omega. \quad (16$$

At this point, we recall that in quantum mechanics,

$$\gamma = \hbar^2/m.$$

Finally, we remark this last equation (16) is a kind of relativistic Schrödinger-Klein-Gordo equation.

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