SCATTERING FOR NEWTONIAN EVOLUTION

Víctor H. Cortés and Claudio Fernández
Departamento de Matemática - Facultad de Matemáticas
P. Universidad Católica de Chile

1 INTRODUCTION

In the study of many phenomena in the physical world, one is interested in describing the long time behavior of certain observables. That is the case, for example, of the principles that govern the radar, the sonar and the dispersion of light waves.

Scattering Theory deals with the long time behavior of evolutions systems that are subject to interactions. One of the main problem consists in studying this behavior, knowing in advance the corresponding interaction. On the other hands, in practice, it is probably more interesting the so called *inverse problem* , which consists of determining the nature of the interaction from the knowledge of some observations made in the long time scale. This is clear in the examples just mentioned above.

Furthermore, Scattering Theory was principally developed for understanding the laws and principles present in Quantum Mechanics. Several quantum experiments can be described in the context of scattering theory, since the interaction one expects to describe is inaccessible, not because it is distant; like in the case of radar, but because it is to small, usually the size of atoms and molecules. Also from the mathematical point of view, the development of scattering theory has been successfully achieved because of the advances on Operator Theory in the last decades.

The Schrödinger equation describes the evolutions in Quantum Mechanics. It can be written in an abstract context as

$$i\frac{\partial u}{\partial t} = Hu, \tag{0.1}$$

where H represents an operator acting in some Hilbert space $\mathcal{H}$ which is a self-adjoint perturbation of a free self-adjoint operator H_0. The usual example is $H_0 = -\Delta$, the spatial laplacian in R^n and $H = -\Delta + V$, for some function V which acts by multiplication in the Hilbert space $L^2(R^n)$.

The corresponding evolution equation in classical mechanics is Newton's equation

$$\frac{d^2x}{dt^2} = F(x,t), \; x \in R^n \tag{0.2}$$

which describes the movement of a particle in R^n with position x under the influence of a force field $F(x,t)$.

These notes are an attempt to show the main ideas and some techniques used in Scattering Theory, avoiding the complexity of quantum mechanics and its mathematical formalism. We shall do it by working directly with the classical Newton's equation, and we will show that these ideas became fruitful and interesting also in this classical framework.

From the point of view scattering theory, the study of the long time behavior of the solutions of equation (0.2) is achieved by comparing it with the solutions of *certain free equation.* It is natural to consider as free equation the one where there are no forces acting on the particle, that is, $F = 0$.

In others words, we shall compare the asymptotical behavior of the solutions of (0.2) with the corresponding behavior of the solutions of the free equation $\ddot{x} = 0$, when t goes to infinity.

1.1 PRELIMINARIES AND NOTATIONS

Under adequate conditions on the force $F(x,t)$, the equations (0.2) has a unique global solution with given initial position in t_0, $x(t_0) = a$ and initial velocity $x'(t_0) = b$. We denote this unique solution as $x(t, t_0; a, b)$ and we simply write $x(t; a, b)$ when $t_0 = 0$. We remark that we have simplified the form of Newton equation by choosing adequate physical units.

For example, the solution of the free equation $\ddot{x} = 0$, with initial conditions $x(t_0) = a$, $x'(t_0) = b$, it is given by $x(t, t_0; a, b) = a + b(t - t_0)$.

It is common to describe the movement of a classical particle using the concept of state. With the above notation the initial *state* is the pair (a, b) and the state at the time t is $(x(t; a, b), x'(t; a, b))$. The collection of all states is called the *phase space* .

In chapter 2 we introduce the concepts of this theory and we discuss the direct problem in dimension $n = 1$ and $n = 3$. Finally, in Chapter 3 we face up the corresponding inverse problem.

2 DIRECT PROBLEM

We denote by Σ the phase space $R^n \times R^n$ and we call Σ_0 the subset of Σ of states with nonzero velocity, that is:

$$\Sigma_0 = \{(x, y) \in \Sigma \ / \ y \neq 0\}.$$

Definition 1 *The wave operators* $W_\pm$ *are functions defined on* Σ_0 *by* $W_\pm(q, p) = (x_\pm, y_\pm)$ *if and only if*

$$\lim_{t \to \pm\infty} \|x(t; x_\pm, y_\pm) - (q + pt)\| = 0$$

$$\lim_{t \to \pm\infty} \dot{x}(t; x_\pm, y_\pm) = p$$

In other words, the solutions $x(t; x_\pm, y_\pm)$ of (0.2) with initial data $(x_\pm, y_\pm)$ behave as the solution $q + pt$ of the free equation $\ddot{x} = 0$, as t approaches $\pm\infty$.

Definition 2 *We say that the wave operators $W_\pm$ for system (0.2) are asymptotically complete if Ran $W_- =$ Ran W_+, where Ran $W_\pm$ denote the image of the transformations $W_\pm$.*

According to this definition, if the wave operators $W_\pm$ are complete then for a given state $(x_+, y_+) \in$ Ran $W_+ =$ Ran W_- there exist states $(q, p), (a, b) \in \Sigma_0$, such that

$$W_+(a, b) = (x_+, y_+) = W_-(q, p)$$

Therefore, if the wave operators are asymptotically complete we can define the following map,

Definition 3 *The scattering operator S is defined as*

$$S = W_+^{-1} W_-$$

Its domain $D(S)$ is a subset of Σ_0 which differs from Σ_0 by a set of Lebesgue measure zero, see [8] for details.

In other words, $S(q, p) = (a, b)$ if and only if there exists a unique solution $x(t)$ of the equation (0.2) satisfying

$$\lim_{t \to -\infty} \dot{x}(t) = p \ , \quad \lim_{t \to -\infty} (x(t) - pt) = q \tag{0.3}$$

and

$$\lim_{t \to +\infty} \dot{x}(t) = b \ , \quad \lim_{t \to +\infty} (x(t) - bt) = a. \tag{0.4}$$

That is, the perturbed trajectory $x(t)$ is asymptotic to $q + pt$, as t approaches $-\infty$ and $x(t)$ is asymptotic to $a + bt$ as t approaches to $+\infty$. We denote (0.3) and (0.4) shortly as $x(t) \sim q + pt$ and $x(t) \sim a + bt$ respectively.

Throughout this notes we shall assume that $V(x) \geq 0$.

2.1 AUTONOMOUS CASE

The autonomous case corresponds to a trajectory $x(t)$ in R^n of a classical particle which it is governed by the Newton's equation

$$\ddot{x} = F(x) \tag{0.5}$$

where the force F acting on the particle depends only on the position of it. We shall deal with the conservative case, that is, we assume that there exists a continuous differentiable potential $V(x)$ such that $V(x) \to 0$ as $\|x\| \to \infty$ and $F(x) = -\nabla\, V(x)$.

Under this force F the system (0.5) is conservative, in other words, the energy E of a solution $x(t)$ of the equation (0.5)

$$E = \frac{1}{2}\|\dot{x}(t)\|^2 + V(x(t)) \tag{0.6}$$

is constant on time, which can be directly proved by showing that the derivative with respect to t of E is zero.

Throughout these notes we shall assume that the potential $F = -\nabla V(x)$ satisfies the following conditions:

C1 There exist $C > 0$ and $\epsilon > 0$ such that for all $x \in R^n$,

$$\|F(x)\| \leq \frac{C}{(1 + \|x\|)^{2+\epsilon}}$$

C2 $F(x)$ is locally Lipschitz and continuous in R^n

C3 There exist $r > 0$, $C > 0$ and $\delta > 0$ such that

$$\|F(x) - F(y)\| \leq \frac{C}{r^{2+\delta}}\|x - y\|$$

for all x, y verifying $\|x\|$, $\|y\| \geq r$.

The hypothesis (C1) and (C2) assure the existence and uniqueness of global solutions of equation (0.5) with prescribed initial conditions $x(t_0) = q$, $\dot{x}(t_0) = p$.

The proof of the next result can be found in [8], and it assures the asymptotic completeness of waves operators for system (0.5).

Theorem 2.1 If $F = -\nabla$ satisfies conditions (C1),(C2) and (C3) then W_+ and W_- exist and they are complete, that is, Ran $W_+ =$ Ran W_-.

We mention that similar result about existence has been studied in [3] for the non autonomous case (0.2). Also, a Hilbert space aproaching has been discussed in [2]. Also [4],[5],[6], [7] and [8]are important references about this subject.

2.2 THE DIRECT PROBLEM IN DIMENSION ONE

Now we will find an explicit formula for S in dimension $n = 1$, for the autonomous case generated by equation (0.5), that is, $F(x) = -V'(x)$.

Let us begin with the following result,

Lemma 2.1 Assume that $(q,p) \in D(S)$ and $S(q,p) = (a,b)$.

a) If $p^2 > 2\|V\|_\infty$ then $b = p$

b) If $p^2 < 2\|V\|_\infty$ then $b = -p$.

Proof Since $V(x)$ converges to zero as $|x|$ approaches infinity, we get from the energy identity (0.6) that $2E = b^2 = p^2$. Assume that $p^2 > 2\|V\|_\infty$ and $p > 0$. Suppose that $b = -p$, then it is clear that $x(-\infty) = -\infty$ and $x(+\infty) = -\infty$. Therefore, there exists $t_0 \in R$ such that $\dot{x}(t_0) = 0$, which contradicts (0.6). A similar argument is applied to the case $p < 0$. Thus proving statement a).

Assume now that $p^2 < 2\|V\|_\infty$ and $p > 0$. If $b = p$, then $x(-\infty) = -\infty$ and $x(+\infty) = +\infty$ .Therefore , there exists $t_0 \in R$ such that $V(x(t_0)) = \|V\|_\infty$. Again from conservation of energy (0.6) , we obtain that $(\dot{x}(t_0))^2 + 2\|V\|_\infty = p^2$ which contradicts our assumption. The case $p < 0$ is treated in a similar way.

Let us define the following sets, which will play an important role in the characterization of the domain of the scattering operator S .

$$E_- = \{x \in R / \ V'(x) = 0 \ and \ V(x) < V(y) \text{ , for all} y < x\}$$

$$E_+ = \{x \in R / \ V'(x) = 0 \ and \ V(x) < V(y) \text{ , for all } y > x\}.$$

That is, E_- (E_+) consists of the local maxima of $V(x)$ which can be seen when one looks from $-\infty$ ($+\infty$). Clearly both $V(E_-)$ and $V(E_+)$ are measurable sets with Lebesgue measure zero.

Theorem 2.2(Characterization of the domain of S)

a) Let $p > 0$. Then $(q,p) \in D(S)$ if and only if $p^2/2 \notin V(E_-)$

b) Let $p < 0$. Then $(q,p) \in D(S)$ if and only if $p^2/2 \notin V(E_+)$.

In particular, the set $\Sigma_0 - D(S)$ has Lebesgue measure zero.

Proof Assume that $(q,p) \in D(S)$ and $p > 0$ .We may assume that $p^2 < 2\|V\|_\infty$. Set $(a,b) = S(q,p)$ and let $x(t)$ be the solution of equation (0.5) which is asymptotic to $q + pt$ as $t \to -\infty$ and asymptotic to $a + bt$ as $t \to +\infty$.

By Lemma 2.1 ,we have that $b = -p$ and therefore $x(-\infty) = -\infty$ and $x(+\infty) = -\infty$. Hence, there exists $t_1 \in R$ such that $\dot{x}(t_1) = 0$.

If $x_0 \in E_-$ is such that $p^2 = 2V(x_0)$ then from (0.6) we conclude that $V(x(t_1)) = V(x_0)$ and therefore $x(t_1) \geq x(t_0)$. Since $x(-\infty) = -\infty$, there exists $t_2 \in R$ such that $x(t_2) = x_0$. Hence $V(x(t_2)) = V(x_0)$. By (0.6) we conclude that $\dot{x}(t_1) = 0$. Finally, by uniqueness of the initial value problem for equation (0.5), we have that $x(t) = x_0$ for all t, which is a contradiction.

In order to prove the reciprocal, let us take p be a positive constant such that $p^2/2 \notin V(E_-)$. Set $(x,y) = W_-(q,p)$ and $x(t) = x(t;x,y)$. Suppose first that $p^2 > 2\|V\|_\infty$. Then, by conservation of energy, we have that

$$(\dot{x}(t))^2 = p^2 - 2V(x(t)) \geq p^2 - 2\|V\|_\infty,$$

which shows that $x(t)$ is unbounded as t converges to $+\infty$. By the argument given in [6], this is enough to conclude that $(x, y) \in W_+$ and so $(q, p) \in D(S)$.

Now, let us consider the case $p^2 < 2\|V\|_\infty$ and let $x_0 \in E_-$ be the smallest point in this set such that $p^2 < 2V(x_0)$. Then, there exists $x_1 < x_0$ verifying $p^2 = 2V(x_1)$. Clearly we have that $V'(x_1) > 0$ and $V(y) < V(x_1)$, for all $y < x_1$.

We claim that there exists $t_1 \in R$ such that $x(t_1) = x_1$. Suppose that this is false.

Since $x(-\infty) = -\infty$ we would have that $x(t) < x_1$, for all $t \in R$. Hence $V(x(t)) < V(x_1)$, for all $t \in R$. By conservation of energy, we have that $\dot{x}(t) \neq 0$, for all $t \in R$.

Since $\dot{x}(-\infty) > 0$, it follows that $x(t)$ is a strictly increasing function of t and therefore $x(\infty) = \lim_{t \to +\infty} x(t)$ exists, $\dot{x}(\infty) = 0$ and $x(-\infty) = 0$ and $x(\infty) \leq x_1$. It follows that $x = x_1$ and therefore $\ddot{x}(\infty) = -V'(x_1) < 0$, which contradicts the fact that $\dot{x}(\infty) = 0$, thus proving our claim.

From (0.6) we have that $\dot{x}(t_1) = 0$. By uniqueness of the solution for (0.5) with given initial data, we deduce that $x(t_1 - t) = x(t_1 + t)$, for all $t \in R$. This shows that $(x, y) \in \text{Ran } W_+$ and so $(q, p) \in D(S)$. A similar argument proves b).

Let us denote, for $p^2 > 2V(x)$, $A(x, p) \equiv A_V(x, p)$ the expression

$$A(x,p) = \frac{p}{\sqrt{p^2 - 2V(x)}} - \frac{p}{|p|}. \tag{0.7}$$

Also, we write for $p^2 < 2\|V\|_\infty$

$$\alpha(p) = \alpha_V(p) \;=\; inf\{x \in R \; / p^2 = 2V(x)\} \;\; if \;\; p > 0,$$

$$\beta(p) = \beta_V(p) \;=\; inf\{x \in R \; / p^2 = 2V(x)\} \;\; if \;\; p < 0.$$

Now, with the notation just introduced we state the following result, which gives the formula for the first component of S.

Lemma 2.2 Let $(q,p) \in D(S)$ and $(a,b) = S(q,p)$.

i) If $p^2 > \|V\|_\infty$ then

$$a = q - \int_{-\infty}^{+\infty} A(x,p)\, dx$$

ii) If $p^2 < 2\|V\|_\infty$ and $p > 0$ then

$$a = -q + 2\alpha(p) + 2\int_{-\infty}^{\alpha(p)} A(x,p)\, dx$$

iii) If $p^2 < 2\|V\|_\infty$ and $p < 0$ then

$$a = -q + 2\beta(p) + 2\int_{\beta(p)}^{\infty} A(x,p)\, dx.$$

Proof We first prove i).

Let us assume that $p^2 > 2\|V\|_\infty$ and $p > 0$ (the case $p < 0$ can be treated in a similar way). Then $\dot{x}(t) \neq 0$ for all t and since $\dot{x}(-\infty) > 0$ we have that $\dot{x}(t) > 0$, for all t. By solving equation (0.6) and integrating in $[t_0, t]$ we obtain that

$$t - t_0 = \int_{x(t_0)}^{x(t)} \frac{dx}{\sqrt{p^2 - 2V(x)}}. \tag{0.8}$$

This last identity implies the following relation:

$$\begin{aligned} -(x(t) - bt) + (x(t_0) - pt_0) &= -x(t) + x(t_0) + b(t - t_0) \\ &= \int_{x(t_0)}^{x(t)} A(x,p)\, dx\ , \end{aligned}$$

which proves assertion i) by taking $t \to \infty$ and $t_0 \to -\infty$.

Let us prove ii).

Suppose that $p^2 < 2\|V\|_\infty$ and $p > 0$. By Lemma 2.1, $x(-\infty) =$

$-\infty$

$x(\infty) = -\infty$, $\dot{x}(-\infty) = p > 0$ and $\dot{x}(\infty) = -p < 0$. Hence, there exists $t_0 \in R$ such that $\dot{x}(t_0) = 0$ and, as in the proof of Theorem 2.2, it follows that $x(t_0 - t) = x(t_0 + t)$, for all $t \in R$. In particular , such t_0 is the unique critical point of $x(t)$. By symmetry, we can write

$$\begin{aligned} x(t) - pt &= x(2t_0 - t) + bt \\ &= x(2t_0 - t) - b(2t_0 - t) + 2bt_0. \end{aligned}$$

Therefore,

$$q = a + 2bt_0 = a - 2pt_0.$$

On the other hand, for $t \leq t_0$ we have that the time $t_0 - t$ is given by (0.8).Then, it follows that

$$(x(t) - pt) - (x(t_0) - pt_0) = \int_{x(t)}^{x(t_0)} A(x,p)\, dx$$

and since $x(-\infty) = -\infty$ and $a = q + p\, t_0$ we finally arrive to

$$a = -q + 2x(t_0) + \int_{-\infty}^{x(t_0)} A(x,p)\, dx.$$

It remains to prove that $x(t_0) = \alpha(p)$. Conservation of energy shows that $2V(x(t_0)) = p^2$ and therefore $\alpha(p) \leq x(t_0)$. On the other hand, since $x(-\infty) = -\infty$, we have that $\alpha(p) = x(t_1)$ for some $t_1 \leq t_0$. Again from (0.6), we conclude that $\dot{x}(t_1) = 0$, which implies that $t_1 = t_0$.

The case $p < 0$ and assertion iii) are proven in a similar fashion. We summarize the above results in the next theorem.

Theorem 2.3 Suppose that $(q,p) \in D(S)$.

a) If $p^2 > 2\|V\|_\infty$, then

$$S(q,p) = (q - \int_{-\infty}^{\infty} A(x,p)\, dx \ \ , \ p)$$

b) If $p^2 < 2\|V\|_\infty$ and $p > 0$, then

$$S(q,p) = (-q + 2\alpha(p) + 2\int_{-\infty}^{\alpha(p)} A(x,p)\, dx\ ,\ -p)$$

c) If $p^2 < 2\|V\|_\infty$ and $p < 0$, then

$$S(q,p) = (-q + 2\beta(p) + 2\int_{\beta(p)}^{\infty} A(x,p)\,dx\ ,\ -p).$$

Corollary 2.1 Suppose that $(q,p) \in D(S)$.

a) If $p^2 > 2\|V\|_\infty$, then $S(q,p) = (q,p) - (B'(p), 0)$, where

$$B(p) = B_V(p) = \int_{-\infty}^{\infty} (\sqrt{p^2 - 2V(x)} - |p|)\,dx$$

b) If $p^2 < 2\|V\|_\infty$, then $S(q,p) = -(q,p) + 2(D'(p)\ ,\ 0)$, where

$$D(p) = \int_{-\infty}^{\alpha(p)} (\sqrt{p^2 - 2V(x)} - p)\,dx\ + p\alpha(p),\ \ for\ \ p > 0$$

and

$$D(p) = \int_{\beta(p)}^{\infty} (\sqrt{p^2 - 2V(x)} + p)\,dx\ + p\beta(p),\ \ for\ \ p < 0$$

Proof By conditions (C1) and (C2), $V \in L^1(R)$, and therefore,

$$\int_{-\infty}^{\infty} (\sqrt{p^2 - 2V(x)} - |p|)\,dx \le \frac{2}{p}\int_{-\infty}^{\infty} V(x)\,dx\ < \infty.$$

By the Lebesgue dominated convergence theorem, the derivatives of $B(p)$ and $D(p)$ exist and a straightforward computation proves the corollary.

Corollary 2.2 Assume that $p^2 > 2\|V\|_\infty$; then

$$S(q,p) = (q,p) - \frac{p}{|p|}(\sum_{k=1}^{\infty} a_k \frac{1}{p^{2k}}\ ,0),$$

where $a_1 = \int_{-\infty}^{\infty} V(x)\,dx$, and $a_k = 1 \cdot 3 \cdots (2k-3)\int_{-\infty}^{\infty} (V(x))^k\,dx$ for $k \ge 2$. Moreover, the series converges uniformly in any neighborhood of $p = \infty$.

Proof It follows by expanding $\sqrt{1 - 2V/p^2} - p/|p|$ in a power series of $1/p$ in a neighborhood of $p = \infty$.

2.3 THE DIRECT PROBLEM IN THREE DIMENSIONS

In this section we will construct the operator S for the three dimensional conservative, radial case.

The radial case takes place when the potential $F(x) = -\nabla V(x)$ in (0.5) depends only on the distance of the particle to the origin; in other words, if we denote by $r = |x|, x \in R^n$, Newton's equation (0.5) becomes

$$\ddot{x} = -\nabla V(r) = -\frac{V'(r)}{r}x \tag{0.9}$$

Let us introduce the following notation: $r = |x|$, $x \in R^3$, $p(t) = \dot{x}(t)$, $J^\wedge = x \wedge p$ (cross product), $J = |J^\wedge|$ and $x \cdot p$ (inner product). It is well known that

$$J^2 = r^2|p|^2 \sin^2 \alpha = r^2(|p|^2 - (\dot{r})^2) = |x|^2|p|^2 - x \cdot p \tag{0.10}$$

where α represents the angle between the vectors x and p.

We assume that $F(x) = \frac{-V'(r)}{r}x$ satisfies conditions (C1),(C2) and (C3), so the asymptotic completeness of wave operators is guaranteed and the scattering operator S can be defined.

In this section we use the notation $S(a_-, b_-) = (a_+, b_+)$ which means , by definition of S, that there exists a unique solution $x(t)$ of (0.9) such that $x(t) \sim a_- + b_-t$ at $t = -\infty$ and $x(t) \sim a_+ + b_+t$ at $t = +\infty$, or equivalently,

a) $x(t) - b_-t$ converges to a_-, as $t \to -\infty$. Similar for $+\infty$ with a_+, b_+.

b) $p(t) = x'(t)$ converges to b_- as $t \to -\infty$. Similar for $+\infty$ with a_+, b_+.

Proposition 2.1 : Let x(t) be the solution of (0.9) such that $x(t) \sim a_- + b_-t$ and $x(t) \sim a_+ + b_+t$. Then

$$J^\wedge = \lim_{t \to -\infty} J^\wedge(t) = a_- \wedge b_- = a_+ \wedge b_+.$$

Proof We write $J^\wedge(t)$ as follows:

$$J^\wedge(t) = (x(t) - b_-t) \wedge p(t) + t(b_- \wedge p(t))$$

Therefore it suffices to prove that the limit as t goes to $-\infty$ of

$t(b_- \wedge p(t))$ is zero.

By integrating equation (0.9) in the interval $]-\infty\ ,\ t\ ]$ we obtain

$$\dot{x}(t) = p(t) = b_- + \int_{-\infty}^{t} (-V'(r)) \frac{x(s)}{r(s)} ds.$$

Thus, by using (0.10) it is easy to check that there exists a positive constant K such that

$$|b_- \wedge p(t)| \leq K \int_{-\infty}^{t} |V'(r(s))| ds.$$

Also there exists $|T| > 0$ sufficiently large such that $|x(s)| \geq M\,|s|$ for any $|s| > |T|$. By condition (C3) it follows that there exist $\epsilon > 0$ and $R > 0$ such that for all $r > R$

$$|V'(r)| \leq \frac{1}{r^{2+\epsilon}}.$$

So for all t with $|t| > |T|$ we obtain that there exists $M > 0$ such that

$$|t|\,|b_- \wedge p(t)| \leq M\,\frac{1}{|t|^{\epsilon}}.$$

The assertion now follows immediately by taking the limit in the last inequality. Finally the identity $a_- \wedge b_- = a_+ \wedge b_+$ is obvious.

The energy of system (0.9) is given by

$$E = \frac{1}{2}|p|^2 + V(r) \tag{0.11}$$

which is constant with respect to time and since $V(r)$ goes to zero as t goes to $\pm\infty$ we have that

$$E = \frac{|b_-|^2}{2} = \frac{|b_+|^2}{2}.$$

Also it follows directly from $r^2 = x \cdot x$ that $r(t) = |x(t)|$ satisfies the ordinary differential equation

$$(\dot{r})^2 + r\ddot{r} = 2(E - V(r)) - r\,V'(r). \tag{0.12}$$

This last equation enables us to prove the following results about the behavior of r.

Proposition 2.2 If $\dot{r}(t_0) = 0$ for some $t_0 \in]-\infty, \infty[$ then $r(t_0+s) = r(t_0-s)$ for any $s \in]-\infty, +\infty[$.

Proof Since $J^\wedge(t)$ is non zero and constant we have that $r(t)$ is bounded below. Also r satisfies the non singular differential equation (0.12). Consider $\tilde{r}(t) = r(2t_0 - t)$. It is easy to see that $\tilde{r}(t)$ is solution of (0.12) with initial conditions $\dot{\tilde{r}}(t_0) = -\dot{r}(t_0) = 0$ and $\tilde{r}(t_0) = r(t_0)$. Therefore by uniqueness we obtain $r(t) = r(2t_0 - t)$ for any t, and so $t = t_0$ is a symmetry axis for r.

Corollary 2.3 Consider $r_0 = r(t_0)$ with $\dot{r}(t_0) = 0$. Then

a) t_0 is unique

b) $r(t)$ is decreasing on $]-\infty, t_0[$

c) $r(t)$ is increasing on $]t_0, +\infty[$.

The results up to now allow us to conclude that the trajectory $x(t)$ takes place on the plane Π which contains the origin (it is clear that $b_- \in \Pi$ and $J^\wedge \cdot b_- = 0$) and with normal in the direction of $J^\wedge = a_- \wedge b_-$. Moreover there exists a unique point $x(t_0)$ on the trajectory $x(t)$ which minimizes the distance to the origin.

It only remains to compute the time $t = t_0$ where such point is attained. The following lemma, which is easy to prove, gives us the exact value of such time.

Lemma 2.3 Assume $x(t) \sim a_+ + b_+ t$ at $t = +\infty$ and $x(t) \sim a_- + b_- t$ at $t = -\infty$. Then:

$$\lim_{t \to -\infty} (r(t) - |bt|) = -\frac{a_- \cdot b_-}{|b|}$$

$$\lim_{t \to +\infty} (r(t) - |b|t) = -\frac{a_+ \cdot b_+}{|b|}$$

Proposition 2.3 Let t_0 be the critical point of $r(t)$; that is $\dot{r}(t_0) = 0$. Then

$$t_0 = -\frac{a_- \cdot b_-}{|b_-|^2} - \frac{r_0}{|b_-|} - \int_{r_0}^{+\infty} \left(\frac{1}{\sqrt{2(E-V) - J^2 r^{-2}}} - \frac{1}{|b_-|} \right) dr$$

Proof Solving equation (0.11) for $|p|$ and replacing it in (0.10) we obtain, by Corollary 2.3, that

$$\dot{r}(t) = -\sqrt{2(E-V) - J^2 r^{-2}} \tag{0.13}$$

Let us integrate the last identity with $r_0 = r(t_0)$.

$$\int_{r_0}^{r(t)} \frac{dr}{\sqrt{2(E-V) - J^2 r^{-2}}} = t_0 - t.$$

Rewriting the last equality as

$$r(t) - |b_-||t| - r_0 - |b_-|t_0 = \int_{r(t_0)}^{r(t)} \left(1 - \frac{|b_-|}{\sqrt{2(E-V) - J^2 r^{-2}}} \right) dr \tag{0.14}$$

and applying Lemma 2.3 as t goes to $-\infty$ we get the corresponding formula for t_0, concluding the proof.

The next proposition deals with another parameter called the "angle of impact" , which is the angle between the asymptotic lines $L_+ : a_+ + b_+ t$ and $L_- : a_- + b_- t$. Let us denote such angle by α.

Proposition 2.4 Let $r_0 = r(t_0)$ be the critical value of $r(t)$ and denote by θ_0 the angle between $x(t_0)$ and the vector b_-. Then $\alpha = \pi - 2\theta_0$ and θ_0 is given by

$$\theta_0 = \int_{r_0}^{+\infty} \frac{J}{r^2} \frac{dr}{\sqrt{2(E-V) - J^2 r^{-2}}}.$$

Proof Let us denote by $\theta(t)$ the angle between $x(t_0)$ and $x(t)$. By elementary geometric considerations we easily arrive to

$$x(t+\epsilon) \cdot x(t) = r(t+\epsilon) r(t) \cos(\theta(t+\epsilon) - \theta(t))$$
$$= \frac{1}{2} \left[r^2(t+\epsilon) + r^2(t) - |x(t+\epsilon) - x(t)|^2 \right].$$

By the mean value theorem one has

$$2r(t+\epsilon)r(t) = \left(\frac{cos(\dot{\theta}(l)\,\epsilon)-1}{\epsilon^2}\right) = \left(\frac{r(t+\epsilon)-r(t)}{\epsilon}\right)^2 - \left(\frac{x(t+\epsilon)-x(t)}{\epsilon}\right)^2$$

By letting ϵ go to zero and since $\theta(-\infty)=0$ and $x(t) \sim a_- + b_- t$ at $t=-\infty$ we obtain

$$\theta(t_0) = \int_{-\infty}^{t_0} \frac{1}{r}\sqrt{|\dot{x}|^2 - (\dot{r})^2}\,dt.$$

Since $\alpha = \pi - 2\theta_0$ we finish the proof by recalling the identities (0.10) and (0.13).

Theorem 2.4 Let $x(t)$ be a solution of (0.9) having scattered conditions $x(t) \sim a_+ + b_+ t$ at $t=+\infty$ and $x(t) \sim a_- + b_- t$ at $t=-\infty$, and let V be a central, radial potential with gradient ∇V satisfying conditions (C1), (C2) and (C3). Then the scattering operator S is defined by

$$S(a_-, b_-) = (a_+, b_+)$$

with

$$a_+ = -\left(2t_0 + \frac{a_- \cdot b_-}{|b_-|^2}\right) b_+ - \frac{1}{|b_-|^2}\left(|J| \sin\alpha b_- - \cos\alpha (b_- \wedge J^\wedge)\right)$$

$$b_+ = |b_-| \cos\alpha b_- + |b_-| \sin\alpha b_- \wedge J^\wedge$$

where α is the angle of impact given by Proposition 2.4 and t_0 is the critical point of $r(t)$ given by Proposition 2.3.

Proof: By Lemma 2.3 and the identity

$$r(t) - |b|t = r(2t_0 - t) - |b(2t_0 - t)| - 2t_0|b|$$

it follows at once that

$$a_+ \cdot b_+ = -a_- \cdot b_- - 2t_0|b_-|^2. \quad (0.15)$$

Consider the orthonormal system $b_-, b_- \wedge J^\wedge, J^\wedge$,. Since we know the angle α, b_+ becomes

$$b_+ = |b_-| \cos\alpha b_- + |b_-| \sin\alpha v$$

where v is the vector determined by $v = b_- \wedge J^\wedge$.

Also, there exist scalars μ and λ such that a_+ can be expressed in this system as $a_+ = \mu b_- + \lambda v$.

From the relation (0.15) and since $J^\wedge = a_+ \wedge b_+$, we obtain the following system:

$$\begin{cases} -\mu \sin\alpha + \lambda \cos\alpha &= \frac{1}{|b_-|} J \\ |b_-|\mu \cos\alpha + \lambda |b_-| \sin\alpha &= -a_- \cdot b_- - 2t_0 |b_-|^2 \end{cases}$$

which has a solution given by

$$\begin{aligned} a_+ &= \left(-\tfrac{J}{|b_-|}\sin\alpha - \tfrac{a_-\cdot b_-}{|b_-|}\cos\alpha\right) b_- + \left(\tfrac{J}{|b_-|}\cos\alpha - \tfrac{a_-\cdot b_-}{b_-}\sin\alpha\right) v - 2t_0 b_+ \\ &= -\tfrac{J}{|b_-|}\sin\alpha b_- + \tfrac{J}{|b_-|}\cos\alpha v - \tfrac{a_-\cdot b_-}{b_-}\left[\cos\alpha b_- + \sin\alpha v\right] - 2t_0 b_+ \\ &= -\left(2t_0 + \tfrac{a_-\cdot b_-}{|b_-|^2}\right) b_+ - \tfrac{1}{|b_-|^2}\left(J \sin\alpha b_- - \cos\alpha (b_- \wedge J)\right) \end{aligned}$$

ending the proof of the theorem.

3 INVERSE PROBLEM

The inverse problem consists of recovering some properties of the potential $V(x)$ from the knowledge of the scattering operator S. We provide a complete answer for dimension one. One can also obtain similar results for dimension $n = 3$, at least in the case where V is radial. We denote the operator S for the equation (0.5) by S_V.

3.1 THE INVERSE PROBLEM IN DIMENSION ONE

In this secti.on we consider the problem of building up the potential V from the corresponding scattering operator S_V . The next result shows that some properties of V are determined from $S_V(p)$ for large energies p .

Theorem 3.1 Let V and W be two potentials verifying conditions (C1), (C2), (C3) and suppose that $S_V = S_W$ .Then

a) $\|V\|_\infty = \|W\|_\infty$.

b) For any interval I not containing zero,

$$m\{\, x \;/\; V(x) \in I \,\} = m\{\, x \;/\; W(x) \in I \,\}$$

where m denotes the Lebesgue measure supported on R.

Proof The first part is a direct consequence of Theorem 2.3.

On the other hand, Corollary 2.2 implies that

$$\int_{-\infty}^{\infty} (V(x))^n \, dx = \int_{-\infty}^{\infty} (W(x))^n \, dx$$

is satisfied for any positive integer n. Therefore for any polynomial P we have that

$$\int_{-\infty}^{\infty} V(x) P(V(x)) \, dx = \int_{-\infty}^{\infty} W(x) P(W(x)) \, dx \, .$$

Since the closures of the ranges of V and W are compact, we conclude by the Weierstrass theorem that the last identity is also valid if one replace the polynomial P by any continuous function f defined on the closure of the range of V which coincides with the corresponding closure of the range of W .

Let I be an interval such that $0 \notin I$ and let $f(x) = \frac{1}{x}\, \chi_I(x)$, where χ_I denotes the characteristic function of I. Approximating f by continuous functions we obtain statement b).

Remarks

1) If $V(x) = W(x+h)$ for all $x \in R$, then $S_V(p) = S_W(p)$ for p large. However, for $p^2 < 2\|V\|_\infty$ and $p > 0$, we have that $\alpha_V(p) = \alpha_W(p) - h$ so that $S_V(q,p) = S_W(q,p) - (2h, 0)$. Thus, if $S_V = S_W$ then V is not a translate of W .

Also, by considering p small and applying Theorem 3.1 we have that the scattering operator determines the sets $V(E_+)$ and $V(E_-)$.

2) The scattering operator can be defined for a class of potentials much larger than those satisfying the conditions given in section 2.1. In fact, even potentials with singularities are allowed.

Example

Let $V_\lambda = \lambda\ |x|^{-n}$, where $n > 1$ and λ is a positive constant. One can prove that the scattering operator S_λ exists. Moreover, for any $p > 0$ and $q \in R$ we have that,

$$\begin{aligned} S_\lambda(q,p) &= (-q + 2\alpha_\lambda(p) + 2\int_{-\infty}^{\alpha_\lambda(p)} A(x,p)\,dx\ ,\ -p) \\ &= -(q,p) + 2\alpha_\lambda(p)\ (1 + \int_1^\infty (1 - \frac{s^{n/2}}{(s^n-1)^{1/2}})\ ds\ ,\ 0), \end{aligned}$$

where we have made the change of variable $x = s\alpha_\lambda(p)$ and we have written α_λ for α_{V_λ} .

Now is clear that for $n = 2$, the last integral is -1 and therefore one gets $S_\lambda(q,p) = -(q,p)$, for all λ. In other words, the potentials $V_\lambda = \lambda|x|^{-2}$, with $\lambda \in R$, all have the same scattering operator.

On the other hand , for $n > 2$ it is easy to see that

$$S_\lambda(q,p) = -(q,p) + 2\alpha_\lambda(p)\ (C_n\ ,\ 0)$$

where C_n is a positive constant. Since α_λ depends explicitly on λ we conclude the scattering operators are diferent for distinct λ's.

Because of the above example, for the inverse problem, we need to impose a decaying condition stronger than (C1).

(C'1) There exist $C > 0$ and $\epsilon > 0$, such that for all $x \in R$

$$|V(x)| \le \frac{C}{1+|x|^{2+\epsilon}}.$$

We now use Abel's transform to recover the potential V from the scattering operator in any interval $]-\infty, x_0[$ where V is strictly increasing.

Let us write $S_1(p) = D'(p)$ in Corollary 2.1(b) for p small and positive. The next theorem gives the explicit formula for V as a function of S_1.

Theorem 3.2 Let V be a potential satisfying (C'1),(C2) and (C3). Then

$$V^{-1}(y) = \frac{1}{\pi}\int_0^y \frac{S_1(\sqrt{2E})}{\sqrt{E(y-E)}}\, dE,$$

for any y with $0 < y < V(\overline{x})$, where $\overline{x}$ is the smallest critical point of V .

Proof Taking the new variable $E = p^2/2$, we have that

$$S_1(\sqrt{2E}) = V^{-1}(E) + \int_{-\infty}^{V^{-1}(E)} \left(\frac{\sqrt{E}}{\sqrt{E - V(x)}} - 1\right) dx,$$

for any $E < \|V\|_\infty$.

Let $\overline{E} = V(\overline{x})$. Since V is strictly increasing in $]-\infty, \overline{x}[$, by putting $\alpha = V^{-1}$ we obtain for $0 < E < \overline{E}$ the following identity

$$\frac{S_1(\sqrt{2E})}{\sqrt{E}} = \frac{V^{-1}(E)}{\sqrt{E}} + \int_0^E \left(\frac{1}{\sqrt{E-s}} - \frac{1}{\sqrt{E}}\right) \alpha'(s)\, ds.$$

By using Fubini's theorem, we have that

$$\int_0^y \frac{S_1(\sqrt{2E})}{\sqrt{E(y-E)}}\, dE = \int_0^y \alpha'(s) \int_s^y \frac{1}{\sqrt{y-E}}\left(\frac{1}{\sqrt{E-s}} - \frac{1}{\sqrt{E}}\right) dE\ ds + \int_0^y \frac{\alpha(E)}{\sqrt{E(y-E)}}\, dE,$$

for any y, $0 < y < \overline{E}$. Since the inner integral in the right hand side has value two, by integration by parts one gets

$$\int_0^y \frac{S_1(\sqrt{2E})}{\sqrt{E(y-E)}}\, dE = \pi\alpha(y) - 2\lim_{h\to 0} \alpha(h) \arcsin(\sqrt{h/y}\,).$$

By hypothesis, $|\alpha(s)|^{2+\epsilon} \le C$, for s small. Therefore, the above limit is equal to zero.

Above theorem allows us to recover from the scattering operator S, through an explicit formula, the potential $V(x)$, up to the first local maximum of $V(x)$. For example, if the potential has only one maximum, then it can be recovered completely. On the other hand, in [1], the authors provide an explicit example of two potentials $V(x), W(x)$ whose scattering operators S_v and S_W coincide. So, S does not give relevant information between others critical points of V.

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