EXPONENTIAL DECAY OF SOLUTIONS OF A SEMILINEAR LIPSCHITZ PERTURBATION OF KIRCHHOFF-CARRIER WAVE EQUATION

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SUMMARY

In this work we study the existence of global solutions and exponential decay of energy of the mixed problem for perturbed Kirchhoff -Carrier wave equation

$$u'' - M(a(u))\Delta u + F(u) + \gamma u' = f$$

where F is a Lipschitz function

RESUMO

Neste trabalho estuda-se a existência de solução global e o decaimento exponencial da solução do problema misto para a equação perturbada de Kirchhoff - Carrier

$$u'' - M(a(u)\Delta u + F(u) + \gamma u' = f$$

onde F é uma função Lipschitziana

INTRODUCTION

Let Ω be a bounded open domain in $\Re^n$ with smooth boundary Γ. Let $F: \Re \to \Re$ be a real nondecreasing Lipschitz function with $F(0) = 0$.

We consider the initial boundary value problem for Kirchhoff operator

$$(1) \quad \begin{cases} Ku + F(u) + \gamma u' = f & \text{in } \Omega \times [0,\infty[\\ u = 0 & \text{on } \Gamma \times [0,\infty[\\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in } \Omega. \end{cases}$$

where

$$\mathrm{K}u = \frac{\partial^2 u}{\partial t^2} - \mathrm{M}\left(\int_\Omega |\nabla\ u|^2 dx\right)\Delta\ u$$

is the Kirchhoff 's operator , M is a Lipschitz continuons function , $\mathrm{M}(s) \geq m_0 \rangle\ 0$ for all $s \in \Re$, and $\gamma \rangle\ 0$ is a given constant.

The problem (1) has its origin in the mathematical description of the spatial vibrations of an elastic streched string when we supposse the tension, in each point, has only vertical component cf. Kirchhoff [8] and Carrier [5].

The mathematical formulation of this model is

$$(2) \qquad \frac{\partial^2 u}{\partial t^2} - \mathrm{M}\left(\int_\Omega |\nabla\ u(x,t)|^2 dx\right)\Delta\ u = f$$

where M (s) is a real function defined on the positive real numbers $s \geq 0$.

There are a large number of results related to model (2). When n = 1 we have the work of Bernstein [3]. He obtained global existence of solution of (2) considering $\mathrm{M}(s) \geq u_0 \rangle\ 0$. In [18] Pohozaev proved that problem (2) has global solution if we take u_0, u_1 and Γ analitic. In [2] Arosio-Spagnolo, [9] Lions, [1] Arosio-Garavaldi the authors have proved global existence of solution of (2). In [2] and [9] m_0, m_1 are chosen in a regular class of functions.

If the equation (2) is perturbed with the term $-\Delta \dfrac{\partial u}{\partial t}$ there exist results on global solution in t cf. Nishiara [16] and Yamada [19]. In [11] Medeiros - Milla Miranda have proved the existence of global solution and exponential decay considering the damping $(-\Delta)^2 \dfrac{\partial u}{\partial t}$, $0 \langle \alpha \leq 1$.

In [7] Hosoya - Yamada considered a nonlinear perturbation of Kirchhoff operator of the following type:

$$(3) \qquad \begin{cases} \mathrm{K}u + |u|^\rho u + \gamma u' = f & \text{on} \quad Q = \Omega\ \text{x}\ [0,\infty[\\ u = 0 \quad \text{on} \quad \Sigma = \Gamma\ \text{x}\ [0,\infty[\\ u(x,0) = u_0(x)\ , \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega \end{cases}$$

where Ku is the Kirchhoff operator.

Considering the perturbation $|u|^\rho u$ and linear damping $\gamma\ u'$, the authors proved the existence of global solution of (3) and studied decay properties, when some smalness condition on u_0, u_1 are considered.

Motived by Hosoya - Yamada work's we consider problem (1) in the case that $F:\Re\to\Re$ is a Lipschitz function. In our previus work we have proved that when $\gamma = 0$, problem (1) has local solution. Our interest now is to obtain global solution of (1) when $\gamma > 0$, and the decay of energy putting some smalness conditions on initial data u_0 and u_1.

2. NOTATION

We consider Ω a regular bounded open set in $\Re^n$, with boundary Γ of class C^2.

By $H^1(\Omega)$ we represent the Sobolev space with the inner product

$$((u,v)) = \int_\Omega u^2 dx + \int_\Omega \nabla u.\nabla v\, dx$$

and norm

$$\|u\|^2 = \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2\, dx$$

By $H_0^1(\Omega)$ we represent the closure of $D(\Omega)$ in $H^1(\Omega)$. In $H_0^1(\Omega)$ we considere the Dirichlet bilinear form

$$a(u,v) = \int_\Omega \nabla u.\nabla v\, dx$$

wich is a inner product, and in $H^1(\Omega)$ we consider the equivalent norm

$$\|u\|^2 = \int_\Omega |\nabla u|^2\, dx.$$

By the same way we define the Sobolev space $H^2(\Omega)$, and prove that the norms of $H_0^1(\Omega)\cap H^2(\Omega)$ and that defined by the Laplace operator are equivalents. See the references Lions-Magenes [10] and Mizohata [13].

3- GLOBAL EXISTENCE OF SOLUTIONS

We suppose that $F:\Re\to\Re$ satisties the following conditions:

(3.1) F is a nondecreasing Lipschitz function with $F(0) = 0$.

We suppose in addition that

(3.2) M is a Lipschitz function, $M(s) \geq m_0 > 0$.

(3.3) $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f \in L^1(0,\infty;\ H_0^1(\Omega) \cap L^2(0,\infty;L^2(\Omega))$

We have the following result.

THEOREM 3.1. Under assumptions (3.1) - (3.3), there exists a positive constant C_0 such that if u_0, u_1, f satisfies

$$|\Delta u_0| + |\nabla u_1| + \int_0^\infty |\nabla f| dt \ \langle \ C_0$$

then there exists a unique function u in the class

(3.4) $u \in C([0,\infty[; H_0^1(\Omega)) \cap C_w([0,\infty[; H_0^1(\Omega) \cap H^2(\Omega))$

(3.5) $u' \in C([0,\infty[; L^2(\Omega)) \cap C_w([0,\infty[; H_0^1(\Omega))$

(3.6) $u'' \in L^2([0,\infty[; L^2(\Omega))$

such that

(3.7) $u(0) = u_0, \qquad u'(0) = u_1$, and

(3.8) $u'' - M(a(u))\Delta u + F(u) + \gamma u' = f \quad$ a.e. in $\Omega \ [0,\infty[$

PROOF - We consider the eigenvalue problem

$$((w_j, v)) = \lambda_j (w_j, v) \text{ for all } v \in H_0^1(\Omega).$$

We have that $w_j \in H_0^1(\Omega) \cap H^2(\Omega)$ and we take $(w_j)_{j \in N}$ orthonormalized in $L^2(\Omega)$. Then $(w_j)_{j \in N}$ is a complete orthonormal system in $L^2(\Omega)$.

Denote by V_m the subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ generated by the first m eigenvectors

(w_j), *i, e,* $\qquad V_m = [w_1, w_2, \ldots, w_m]$

Let $u_m(t) \in V_m$ defined by

(3.9) $\quad (K(u_m(t)), v) + (F(u_m(t)), v) + \gamma (u_m'(t), v) = (f(t), v)$ for all $v \in V_m$.

(3.10) $u_m(0) = u_{0m} \to u_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$.

(3.11) $u_m'(0) = u_{1m} \to u_1$ in $H_0^1(\Omega)$.

The system (3.9) with initial conditions (3.10) and (3.11) has local solution on $[0, t_m[$. The extention to $[0,T[$ is consequence of the first estimate.

ESTIMATE (i) - Let us consider $v = 2u_m'(t)$ in (3.9) we obstain

$$(3.12)\quad \frac{d}{dt}\left|u_m'(t)\right|^2 + M\big(a(u_m(t))\big)\frac{d}{dt}a(u_m(t)) + 2\int_\Omega F(u_m(t))\,u_m'(t)dx +$$
$$+\ 2\gamma\left|u_m'(t)\right|^2 = 2\big(f(t),\, u_m'(t)\big)$$

If G is a primitive of F, that is $G(\lambda) = \int_0^\lambda F(s)ds$ we have

$$\frac{d}{dt}G(u_m(t)) = F(u_m(t))u_m'(t)$$

Also, if $\hat{M}(\lambda)$ is a primitive of M, it follows that

$$\frac{d}{dt}\hat{M}\big(a(u_m(t))\big) = M\big(a(u_m(t))\big)\frac{d}{dt}a(u_m(t))$$

Modifying (3.12) we obtain

$$(3.13)\quad \frac{d}{dt}\left[\left|u_m(t)\right|^2 + \hat{M}\big(a(u_m(t))\big) + 2\int_\Omega G(u_m(t))\,dx\right] + 2\gamma\left|u_m(t)\right|^2 \le \frac{1}{\gamma}\left|f(t)\right|^2 + \gamma\left|u_m'(t)\right|^2$$

where $\gamma \rangle 0$ is a constant.

Integrating (3.14) from 0 to $t \langle t_m$ we get.

$$(3.14)\quad \left|u_m'(t)\right|^2 + \hat{M}\big(a(u_m(t))\big) + 2\int_\Omega G(u_m(t))\,dx + \gamma\int_0^t\left|u_m'(s)\right|^2 ds \le$$
$$\le \left|u_{1m}\right|^2 + \hat{M}\big(a(u_{0m})\big) + 2\int_\Omega G(u_{0m})\,dx + \frac{1}{\gamma}\int_0^\infty\left|f(s)\right|^2 ds$$

By (3.10) $u_{0m} \to u_0$ in $H_0^1(\Omega)\cap H^2(\Omega)$.

There exists a sus sequence $(u_{0m})_{u\in N}$ such that $u_{0m} \to u_0$ a. e in Ω. Then G u_{0m} converge to G u_0 a. e in Ω. Since F is a Lipschitz function we have

$$\left|G(u_{0m}(x))\right| = \left|\int_0^{u_{0m}(x)} F(s)ds\right| \le \int_0^{u_{0m}(x)} F(s).F(0)\,ds \le C\left|u_{0m}(x)\right|^2$$

By Lebesgue Dominated convergence Theorem we get

$$\int_\Omega\left|G(u_{0m}(x))\right|dx \to \int_\Omega\left|G(u_0(x))\right|dx$$

By the same argument

$$\hat{M}(a(u_{0m})) \to \hat{M}(a(u_0))$$

It follows that

$|u_{1m}|^2 + \hat{M}(a(u_{0m})) + 2\int_\Omega G(u_{0m}(x))\, dx + \frac{1}{\gamma}\int_0^\infty |f(s)|^2\, ds$ is bounded independent of m. Representing by C a constant such that

$$(3.15)\quad |u_{1m}|^2 + \hat{M}(a(u_{0m})) + 2\int_\Omega G(u_{0m}(x))\, dx + \frac{1}{\gamma}\int_0^\infty |f(s)|^2\, ds < C$$

we have

$$(3.16)\quad |u_m'(t)|^2 + \hat{M}(a(u_m(t))) + 2\int_\Omega G(u_m(t))\, dx + \gamma\int_0^t |u_m'(s)|^2\, ds < C$$

for all $t \geq 0$

By assumption (3.1) we have that $\int_\Omega G(u_m(t))\, dx \geq 0$.

Also, $\hat{M}(a(u_m(t))) = \int_0^{a(u_m(t))} M(s)\, ds \geq m_0\, a(u_m(t))$.

From (3.16) we obtain the estimate (i),

$$(3.17)\quad |u_m'(t)|^2 + |\nabla u_m(t)|^2 < C_1 \quad \text{for all } t \geq 0.$$

ESTIMATE (ii) - Taking $v = -\Delta u_m'(t)$ in (3.9) we have

$$(3.18)\quad \frac{1}{2}\frac{d}{dt}|\nabla u_m'(t)|^2 + M(a(u_m(t)))\frac{d}{dt}|\Delta u_m(t)|^2 + \gamma|\nabla u_m'(t)|^2 \leq$$
$$\leq -(F(u_m(t)), -\Delta u_m(t)) + (f(t), -\Delta u_m'(t))$$

We have that

$$\frac{d}{dt}\left[M(a(u_m(t)))|\Delta u_m(t)|^2\right] = M(a(u_m(t)))\frac{d}{dt}|\Delta u_m(t)|^2 + |\Delta u_m(t)|^2 M'(a(u_m(t)))\frac{d}{dt}a(u_m(t))$$

We modify (3.19) obtaining

$$(3.19)\quad \frac{d}{dt}\left[\left|\left(\nabla u_m'(t)\right)\right|^2+\mathrm{M}\left(a(u_m(t))\right)\left|\Delta u_m(t)\right|^2\right]+2\,\gamma\left|\nabla u_m'(t)\right|^2\le$$
$$\le 2\,\mathrm{M}'\left(a(u_m(t))\right)\left|\nabla u_m(t)\right|^2\left|\nabla u_m'(t)\right|\left|\Delta u_m(t)\right|^2+$$
$$+\,2\left|\left(\nabla\ \mathrm{F}\,(u_m(t));\Delta u_m'(t)\right)\right|+2\left|\left(\nabla f(t),\nabla u_m'(t)\right)\right|$$

The second member is estimated by

$$(3.20)\quad \left|\mathrm{M}'\left(a(u_m(t))\right)\right|\left|\nabla\ u_m(t)\right|\left|\nabla\ u_m'(t)\right|\left|\Delta\ u_m(t)\right|^2\le \mathrm{M}_1\mathrm{C}_1\left|\nabla\ u_m'(t)\right|\left|\Delta\ u_m(t)\right|^2$$

where $\mathrm{M}_1=\max\left\{\left|\mathrm{M}'(\lambda)\right|,\ 0\le\lambda\le\mathrm{C}_1\right\}$ and $\left|\nabla\ u_m(t)\right| \langle\ \mathrm{C}_1$ by estimate (i)

Since F is a Lipschitz function

$$\left|\left(\nabla\ \mathrm{F}(u_m(t)),\ \nabla u_m'(t)\right)\right|=\left|\left(\mathrm{F}'(u_m(t))\ \nabla u_m(t),\ \nabla u_m'(t)\right)\right|\ \le\ \mathrm{C}\left|\nabla u_m(t)\right|\left|\nabla u_m'(t)\right|$$

We have also

$|\Delta u|\ge\sqrt{\lambda_1}\,|\nabla u|$ for all $u\ \in\ \mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)$,

where λ_1 is the least eigenvalue of $-\Delta$.

Therefore

$$(3.21)\quad \left|\left(\mathrm{F}\,(u_m(t)),\ -\Delta u_m'(t)\right)\right|\ \le\ \mathrm{C}\left|\nabla u_m(t)\right|\left|\nabla u_m'(t)\right|\ \le\ \frac{\mathrm{C}^2}{2\lambda_1\gamma}\left|\Delta\ u_m(t)\right|^2\ +\ \frac{\gamma}{2}\left|\nabla\ u_m'(t)\right|^2$$

where $\gamma>0$ is an arbitrary constant.

Substituting (3.20) and (3.21) in to (3.19) we obtain

$$(3.22)\quad \frac{d}{dt}\left[\left|\nabla\ u_m'(t)\right|^2+\mathrm{M}\left(a(u_m(t))\right)\left|\Delta\ u_m(t)\right|^2\right]+\frac{3}{2}\gamma\left|\nabla\ u_m'(t)\right|^2\le$$
$$\le 2\mathrm{M}_1\mathrm{C}_1\left|\nabla\ u_m'(t)\right|\ \left|\Delta\ u_m(t)\right|^2+\frac{\mathrm{C}_2^2}{2\gamma}\left|\nabla\ u_m'(t)\right|^2+2\left|\nabla\ f(t)\right|\ \left|\nabla\ u_m'(t)\right|$$

where $\mathrm{C}_2=\dfrac{\mathrm{C}}{\sqrt{\lambda_1}}\ \rangle\ 0$

Taking $v=-\Delta\ u_m(t)$ in (3.9) we obtain

$$(3.23)\quad \left(\nabla\ u_m''(t),\ \nabla u_m(t)\right)\ +\ \mathrm{M}\left(a(u_m(t))\right)\left|\Delta u_m(t)\right|^2+$$
$$+\left(\mathrm{F}(u_m(t)),\ -\Delta u_m(t)\right)\ +\ \gamma\left(\nabla u_m'(t),\ \nabla u_m(t)\right)=\left(\nabla f(t),\ \nabla u_m(t)\right)$$

We have

$$(3.24)\quad \frac{d}{dt}\left(\nabla u_m'(t),\ \nabla u_m(t)\right) = \left(\nabla u_m''(t),\ \nabla u_m(t)\right) + \left|\nabla u_m'(t)\right|^2$$

and by hipothesis (3.1)

$$(3.25)\quad \left(\mathrm{F}(u_m(t)),\ -\Delta u_m(t)\right) = \left(\nabla\ \mathrm{F}(u_m(t)),\ \nabla u_m(t)\right) =$$
$$= \left(\mathrm{F}'(u_m(t))\Delta u_m(t),\ \nabla u_m(t)\right) = \int_\Omega \mathrm{F}'(u_m(t))\left|\nabla u_m(t)\right|^2 dx\ \geq\ 0$$

From (3.23) and using (3.24) , (3.25) and (3.2) we have

$$(3.26)\quad \frac{d}{dt}\left[\left(\nabla u_m'(t),\ \nabla u_m(t)\right) + \frac{\gamma}{2}\left|\nabla u_m'(t)\right|^2\right] + m_0\left|\Delta u_m(t)\right|^2 \leq$$
$$\leq\ \left|\nabla u_m'(t)\right|^2 + \left|\nabla f(t)\right|\,\left|\nabla u_m(t)\right|$$

Multiplying (3.26) by γ and adding (3.22) we obtain

$$(3.27)\frac{d}{dt}\left[\left|\nabla u_m'(t)\right|^2 + \mathrm{M}\left(a(u_m(t))\right)\left|\Delta u_m(t)\right|^2 + \gamma\left(\nabla u_m'(t),\ \nabla u_m(t)\right) + \frac{\gamma^2}{2}\left|\nabla u_m(t)\right|^2\right] +$$
$$u_0\gamma\left|\Delta u_m(t)\right|^2\ +\ \frac{\gamma}{2}\left|\nabla u_m'(t)\right|^2\ \leq\ 2\mathrm{M}_1\mathrm{C}_1\left|\nabla u_m'(t)\right|\,\left|\Delta u_m(t)\right|^2\ +\ \frac{\mathrm{C}_2^2}{2\gamma}\left|\Delta u_m'(t)\right|^2\ +$$
$$+\ \left|\nabla f(t)\right|\left[2\left|\nabla u_m'(t)\right|\ +\ \left|\nabla u_m(t)\right|\right]$$

We define

$$(3.28)\quad \mathrm{H}(u(t)) = \left|\nabla u'(t)\right|^2 + \mathrm{M}\left(a(u(t))\right)\ \left|\Delta u(t)\right|^2 + \gamma\left(\nabla u'(t),\ \nabla u(t)\right)\ +\ \frac{\gamma^2}{2}\left|\nabla u(t)\right|^2$$

then from (3.27) it follows

$$(3.29)\quad \frac{d}{dt}\mathrm{H}(u_m(t))\ +\ \frac{\gamma}{2}\left|\nabla u_m'(t)\right|^2\ +\left[u_0\gamma - \frac{\mathrm{C}_2^2}{2\gamma}\ -\ 2\,\mathrm{M}_1\mathrm{C}_1\left|\nabla u_m'(t)\right|\right]\left|\Delta u_m(t)\right|^2 \leq$$
$$\leq\ \left|\nabla f(t)\right|\left[\ 2\ \left|\nabla u_m'(t)\right|\ +\ \left|\nabla u_m(t)\right|\right]$$

If $\gamma\ \rangle\ \dfrac{\mathrm{C}_2}{\sqrt{2m_0}}$ then $m_0\gamma\ -\ \dfrac{\mathrm{C}_2^2}{2\gamma}\ \rangle\ 0$

Furthermore

$$\gamma\left(\nabla u_m'(t),\ \nabla u_m(t)\right) \geq -\frac{1}{2}\left|\nabla u_m'(t)\right|^2 - \frac{\gamma^2}{2}\left|\nabla u_m(t)\right|^2$$

then

(3.30) $H(u_m(t)) \le |\nabla u_m'(t)|^2 + u_0|\Delta u_m(t)|^2$

Since $|\Delta u(t)| \ge \sqrt{\lambda_1}|\nabla u(t)|$ for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$

we have

(3.31) $$H(u_m(t)) \le |\nabla u_m'(t)|^2 + M_0|\Delta u_m(t)|^2 + \gamma|\nabla u_m'(t)|\,|\nabla u_m(t)| + \\ + \frac{\gamma^2}{2}|\nabla u(t)|^2 \le \frac{3}{2}|\nabla u_m'(t)|^2 + \left(M_0 + \frac{\gamma^2}{\lambda_1}\right)|\Delta u_m(t)|^2$$

where

$$M_0 = \max\{|M(\lambda)|,\ 0 \le \lambda \le C_1\}$$

We define

$$E^x(m(t)) = |\nabla m'(t)|^2 + |\Delta m(t)|^2$$

then from (3.30) and (3.31) there exist constants $C_3 \rangle 0$ and $C_4 \rangle 0$, independent of *m*, such that

(3.32) $$C_3 E^x(u_m(t)) \le u_m(t) \le C_4 E^x(u_m(t))$$

Inequality (3.32) allows us to deduce a priori estimates for E^x $\hbar_m(t)g$

We take u_{1m} satisfying

(3.33) $$2M_1C_1|\nabla u_{1m}| \le \frac{1}{4}\left(u_0\gamma - \frac{C_2^2}{2\gamma}\right), \quad \gamma \rangle \frac{C_2}{\sqrt{2m_0}}$$

We will show that

(3.34) $$2M_1C_1|\nabla u_m'(t)| \le \frac{1}{2}\left(u_0\gamma - \frac{C_2^2}{2\gamma}\right)$$

for all $0 \le t \le \infty$, putting some additional conditions on $\{u_{0m}, u_{1m}, f\}$.

Suppose that there exists $\xi > 0$, such that (3.34) holds for $0 \le t \langle \xi$, and

$$(3.35)\quad 2M_1C_1\left|\nabla u'(\xi)\right| = \frac{1}{2}\left(u_0\gamma - \frac{C_2^2}{2\gamma}\right),\quad \gamma \rangle\ \frac{C_2}{\sqrt{2m_0}}$$

From (3.29) and using (3.30) and (3.31) we have

$$(3.36)\quad \frac{d}{dt}H(u_m(t)) + 2\gamma\beta\ H(u_m(t)) \leq 2\ C_5\left|\nabla f(t)\right|\ H(u_m(t))^{\frac{1}{2}}$$

for $0 \leq t \leq \xi$, with $C_5\ \rangle\ 0$ and $\beta\ \rangle\ 0$ constants.

Rewrite this differential inequality, we obtaim

$$(3.37)\quad \frac{d}{dt}\left[e^{2\beta\gamma t}\ H(u_m(t))\right] \leq 2\ C_5\ e^{2\beta\gamma t}\left|\nabla f(t)\right|\left[e^{2\beta\gamma t}\ H(u_m(t))\right]^{\frac{1}{2}}$$

Solving (3.37) it results

$$(3.38)\quad H(u_m(t))^{\frac{1}{2}} \leq H(u_m(0))^{\frac{1}{2}}e^{-2\beta\gamma t} + C_5\int_0^t e^{\beta\gamma(s-t)}\left|\nabla f(s)\right|ds \leq$$
$$\leq H(u_m(0))^{\frac{1}{2}} + C_5\int_0^\infty \left|\nabla f(s)\right|ds$$

for $0 \leq t \leq \xi$

From (3.38) and (3.32) it follous

$$(3.39)\quad \left|\nabla u_m'(\xi)\right|^2 \leq E^x\left(u_m(\xi)\right) \leq \frac{1}{C_3}\ H(u_m(\xi)) \leq H_0$$

where $H_0 = H(u_m(0))^{\frac{1}{2}} + C_5\int_0^\infty \left|\nabla f(s)\right|ds$

We make H_0 sufficiently small so that

$$(3.40)\quad 2M_1C_1H_0 \leq \frac{1}{2}\left(u_0\gamma - \frac{C_2^2}{2\gamma}\right),\qquad \gamma \rangle\ \frac{C_2}{\sqrt{2m_0}}$$

holds.

From (3.39) it follows that

$$2M_1C_1\left|\nabla u'(\xi)\right| \langle\ \frac{1}{2}\left(m_0\gamma - \frac{C_2^2}{2\gamma}\right),\quad \gamma \rangle\ \frac{C_2}{\sqrt{2m_0}}$$

which contradicts to (3.35). Therefore we conclude that (3.34) is valid for H_0 sufficiently small.

If m is sufficiently large and $|\Delta u_0|$, $|\nabla u_1|$ and $|\nabla f|_{L^2(0,\infty;L^2(\Omega))}$ are sufficiently small we have that (3.40) and (3.33) are valid.

From (3.34) we conclude that

(3.41) $E^x \quad u_m(t)g \quad C_6$ for all $t \geq 0$, or

(3.42) $\left|\nabla u_m'(t)\right|^2 + \left|\Delta u_m(t)\right|^2 < C_6$ for all $t \geq 0$.

From (3.29), (3.34) and $|\Delta u(t)| \geq \sqrt{\lambda_1}|\nabla u(t)|$ for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$ we obtain

$$\frac{d}{dt}H(u_m(t)) + \frac{\gamma}{2}\left|\nabla u_m'(t)\right|^2 + \frac{1}{2}\left(u_0\gamma - \frac{C_2^2}{2\gamma}\right)\left|\Delta u_m(t)\right|^2 \leq |\nabla f(t)|\left[2\left|\nabla u_m'(t)\right| + \sqrt{\lambda_1}\left|\Delta u_m(t)\right|\right]$$

Using (3.41) we get

$$\text{(3.43)} \quad \frac{d}{dt}H(u_m(t)) + \frac{\gamma}{2}\left|\nabla u_m'(t)\right|^2 + \frac{1}{2}\left(u_0\gamma - \frac{C_2^2}{2\gamma}\right)\left|\Delta u_m(t)\right|^2 \leq C_7|\nabla f(t)|$$

for $\gamma > \dfrac{C_2}{\sqrt{2m_0}}$

Integrating (3.43) from 0 to t, it follows

$$\text{(3.44)} \quad H(u_m(t)) + \frac{\gamma}{2}\int_0^t\left|\nabla u_m'(s)\right|^2 ds + \frac{1}{2}\left(m_0\gamma - \frac{C_2^2}{2\gamma}\right)\int_0^t\left|\Delta u_m(s)\right|^2 ds \leq$$

$$\leq H(u_m(0)) + C_7\int_0^t|\nabla f(s)|ds \leq C_8 \quad \text{for all } t \geq 0,\ C_8 > 0 \text{ a constant.}$$

From (3.44) we obtain the estimate

$$\int_0^t\left|\nabla u_m'(s)\right|^2 ds + \int_0^t\left|\Delta u_m(s)\right|^2 ds < C_9 \quad \text{for all } t \geq 0, \quad C_9 > 0.$$

ESTIMATE (iii) - Taking $v = u_m''(t)$ in (3.9) we obtain

$$\left|u_m''(t)\right|^2 - M(a(u_m(t)))\left(\Delta u_m(t), u_m''(t)\right) + \left(F(u_m(t)), u_m''(t)\right) +$$

$$+ \gamma\left(u_m'(t), u_m''(t)\right) = \left(f(t), u_m''(t)\right)$$

or

$$\left|u_m^{\cdot\cdot}(t)\right|^2 \leq M_0\left|\Delta u_m(t)\right| + \left|F(u_m(t))\right| + \gamma\left|u_m^{\cdot}(t)\right| + \left|f(t)\right|$$

where

$M_0 = \{|M(\lambda)|;\ 0 \leq \lambda \leq C_1\}$.

Since F is a Lipschitz function we have

$$\left|F(u_m(t))\right|^2_{L^2(\Omega)} \leq C\left|F(u_m(t))\right|^2_{L^2(\Omega)} \leq C\left|\nabla\ u_m(t)\right|^2_{L^2(\Omega)},\quad C > 0$$

then

$$\int_0^t\left|u_m^{\cdot\cdot}(s)\right|^2 ds \leq M_0^2\int_0^t\left|\Delta u_m(s)\right|^2 ds + C\int_0^t\left|\nabla u_m(s)\right|^2 ds + \gamma^2\int_0^t\left|u_m^{\cdot}(s)\right|^2 ds + \int_0^t\left|f(s)\right|^2 ds$$

Using estimate (i), (ii) and $f \in L^2(0,\ \infty;\ L^2(\Omega))$ it results

(3.45) $\int_0^t\left|u_m^{\cdot\cdot}(s)\right|^2 ds < C_{10}$ for all $t \geq 0$, $C_{10} > 0$

From estimates (i), (ii), (iii) we obtain

(3.46) $(u_m)_{u \in N}$ bounded in $L^\infty(0,\ T;\ H_0^1(\Omega)\cap H^2(\Omega))$

(3.47) $(u_m)_{u \in N}$ bounded in $L^\infty(0,\ T;\ H_0^1(\Omega))$

(3.48) $(u_m^{\cdot})_{u \in N}$ bounded in $L^\infty(0,\ T;\ H_0^1(\Omega))$

(3.49) $(u_m^{\cdot})_{u \in N}$ bounded in $L^\infty(0,\ T;\ L^2(\Omega))$

(3.50) $(u_m^{\cdot\cdot})_{u \in N}$ bounded in $L^2(0,\ T;\ L^2(\Omega))$

From (3.46) and (3.47) and by Aubin-Lions Compactness Theorem exists a subsequence, still represented by $(u_m)_{u \in N}$,such that

(3.51) $u_m \to u$ strongly in $L^2(0,\ T;\ H_0^1(\Omega))$

Using the same arguments and continuous injections $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ with the imbedding from $H_0^1(\Omega)$ into $L^2(\Omega)$ compact we obtain

(3.52) $u_m \to u$ strongly in $L^2(0,\ T;\ L^2(\Omega))$.

Then

$$\int_0^T \left|F(u_m(s)) - F(u(s))\right|_{L^2(\Omega)} ds \le \int_0^T \left|u_m(s) - u(s)\right|_{L^2(\Omega)} ds \le C\sqrt{T}\left[\int_0^T \left|u_m(s) - u(s)\right|^2_{L^2(\Omega)} ds\right]^{1/2}$$

Since $u_m \to u$ strongly in $L^2(0,\ T;\ H_0^1(\Omega))$ we conclude that

(3.53) $F(u_m) \to F(u)$ strongly in $L^1(0,\ T)$

From (3.50) we have

(3.54) $u_m^{\cdot} \to u^{\cdot}$ weakly in $L^2(0,\ T;\ L^2(\Omega))$.

Then from (3.10)

$$\int_0^T (u_m^{\cdot}(s),\ v)\ \theta(s)ds + \int_0^T M(a(u_m(s)))\, a(u_m^{\cdot}(s),\ v)\ \theta(s)ds + \int_0^T F(u_m^{\cdot}(s),\ v)\ \theta(s)ds +$$
$$+ \gamma\int_0^T (u_m^{\cdot}(s),\ v)\ \theta(s)ds = \int_0^T (f(s),\ v)\ \theta(s)ds$$

for all $v \in H_0^1(\Omega)$ and $\theta \in D(0,\ T)$

Taking the limit when $m \to \infty$ we have

$$\int_0^T (u_m^{\cdot}(s),\ v)\ \theta(s)ds + \int_0^T M(a(u_m(s)))\, a(u(s),\ v)\ \theta(s)ds + \int_0^T F(u(s),\ v)\ \theta(s)ds +$$
$$+ \gamma\int_0^T (u^{\cdot}(s),\ v)\ \theta(s)ds = \int_0^T (f(s),\ v)\ \theta(s)ds$$

or

$$\int_Q (u^{\cdot} - M(a(u))\Delta u + F(u) + \gamma\ u^{\cdot},\ \Phi)ds\,dx = \int_Q f(s)\ \Phi\ ds\,dx \text{ for all } \Phi \in D(Q).$$

By Du Bois Raymond Lemma

$$u^{\cdot} - M(a(u))\Delta u + F(u) + \gamma\ u^{\cdot} = f \quad \text{a. e. in Q.}$$

As a consequence of estimates (i) , (ii) and (iii) we have

$$u \in L^\infty\left(0,\ T;\ H_0^1(\Omega)\cap H^2(\Omega)\right)$$

$$(3.35)\ u' \in L^\infty\left(0,\ T;\ H_0^1(\Omega)\right)$$

$$u'' \in L^2\left(0,\ T;\ L^2(\Omega)\right)$$

therefore the regularity properties (3.4) , (3.5) and (3.6) are consequence of (3.55).

Finally by (3.4) and (3.5) we have $u(0) = u_0,\ \ u'(0) = u_1$.

UNIQUENESS - The uniqueness of solution follows using regularity properties (3.4), (3.5), (3.6) and F Lipshitz.

4- ASYMPTOTIC BEHAVIOUR

On this section we obtain information on the behaviour of the energy associated to problem (1) when t goes to ∞.

THEOREM - 4.1 - If m is the solution of theorem 3.1 them there exist constants $\alpha>0$ and $C>0$ such that

$$(4.1)\quad E(u(t))^{\frac{1}{2}} \le C\left[E(u(0))^{\frac{1}{2}}e^{-\alpha\gamma t} + \int_0^t e^{\alpha\gamma(s-t)}|f(s)|ds\right]$$

for all $t \ge 0$, where

$$(4.2)\quad E(u(t)) = |u'(t)|^2 + \hat{M}(a(u(t))) + 2\int_\Omega G(u(t))dx$$

If

$$(4.3)\quad E^*(u(t)) = |\nabla u'(t)|^2 + |\Delta u(t)|^2$$

then there exist positive constants β and C^* such that

$$(4.4)\quad E^*(u(t))^{\frac{1}{2}} \le C^*\left[E^*(u(0))^{\frac{1}{2}}e^{-\beta\gamma t} + \int_0^t e^{\beta\gamma(s-t)}|\nabla f(s)|ds\right] \text{ for all } t \ge 0, \text{ and}$$

$$(4.5)\quad \lim_{t\to\infty} E(u(t)) = \lim_{t\to\infty} E^*(u(t)) = 0$$

PROOF - Let $u_m(t)$ defined by (3.9) - (3.11). Taking $v = u_m(t)$ in (3.9) we obtain

$$(u_m'(t),\, u_m(t)) + M(a(u_m(t)))a(u_m(t)) + (F(u_m(t)),\, u_m(t)) + \gamma\,(u_m'(t),\, u_m(t)) = (f(t),\, u_m(t))$$

We define

$$(4.7)\quad S(u(t)) = E(u(t)) + \tau\left[2(u'(t),\, u(t)) + \gamma|u(t)|^2\right]$$

where τ is a positive number to be determined.

We have

$$(4.8)\quad \frac{d}{dt}S(u_m(t)) = \frac{d}{dt}E(u_m(t)) + \tau\left[2|u_m'(t)|^2 + 2(u_m''(t),\, u_m(t)) + 2\gamma(u_m(t),\, u_m'(t))\right]$$

By (3.13) we have

$$(4.9)\quad \frac{d}{dt}E(u_m(t)) + 2\gamma|u_m'(t)|^2 = 2(f(t),\, u_m'(t))$$

Multiplying (4.6) by 2τ, adding (4.9) and observing that $s\,F(s) \geq G(s) \geq 0$ where $G(\lambda) = \int_0^\lambda F(s)ds$ it follows

$$(4.10)\quad \frac{d}{dt}S(u_m(t)) + 2(\gamma-\tau)\,|u_m'(t)|^2 + 2\tau u_0|\nabla\, u_m(t)|^2 + 2\tau\int_\Omega G(u_m(t))dx \leq$$
$$\leq 2|f(t)|\,\left[|u_m'(t)| + \tau|u_m(t)|\right]$$

From (4.7) using Poincaré inequality for $u_0 \in H_0^1(\Omega)$ and observing that M a $b_m(t)$ is uniformly bounded on 0, ∞ , there exists a constant $_2$ 0 such that

$$(4.11)\quad S(u_m(t)) \leq {}_2\left[|u_m'(t)|^2 + |\nabla\, u_m(t)|^2 + 2\int_\Omega G(u_m(t))dx\right]$$

On the other hand

$$2\tau(u'(t),\, u(t)) \geq -\frac{\tau}{\gamma}|u'(t)|^2 - \tau\gamma|u(t)|^2$$

then, choosing, $\tau = \frac{\gamma}{2}$, we have

$$(4.12)\quad S(u_m(t)) \geq \left(1-\frac{\tau}{\gamma}\right)|u_m'(t)|^2 + u_0|\nabla\, u_m(t)|^2 + 2\int_\Omega G(u_m(t))dx \geq$$

$$\geq K_3\left[|u_m'(t)|^2 + |\nabla\, u_m(t)|^2 + 2\int_\Omega G(u_m(t))dx\right]$$

Hence, from (4.11) and (4.12) we obtain

$$(4.13)\quad K_3\, E(u_m(t)) \leq S(u_m(t)) \leq K_2\, E(u_m(t))$$

and from (4.10),

$$(4.14)\quad \frac{d}{dt}S(u_m(t)) + 2\alpha\gamma\, S(u_m(t)) \leq 2K_4|f(t)|\, S(u_m(t))^{1/2}$$ with some constants $\alpha > 0$ and $K_4 > 0$.

Solving the differential inequality (4.14) it follows

$$(4.15)\quad S(u_m(t)) \leq C\left[S(u_m(0))^{1/2}e^{-\alpha\gamma\, t} + \int_0^t e^{\alpha\gamma(s-t)}|f(s)|ds\right]$$

Since (4.13) and (4.15) we obtain

$$(4.16)\quad E(u_m(t))^{1/2} \leq C\left[E(u_m(0))^{1/2}e^{-\alpha\gamma\, t} + \int_0^t e^{\alpha\gamma(s-t)}|f(s)|ds\right]$$

Taking the limit when $m \to \infty$ the same manner as in Theorem 3.1, results

$$(4.17)\quad E(u(t))^{1/2} \leq C\left[E(u(0))^{1/2}e^{-\alpha\gamma t} + \int_0^t e^{\alpha\gamma(s-t)}|f(s)|ds\right],\ C > 0.$$

We also have that

$$C_3\, E^*(u_m(t)) \leq H(u_m(t)) \leq C_4\, E^*(u_m(t))$$

then there exists constants $\beta > 0$ and $C^* > 0$ such that

$$(4.18)\quad E^*(u_m(t)) \leq C^*\left[E^*(u_m(0))^{1/2}e^{-\alpha\gamma t} + \int_0^t e^{\alpha\gamma(s-t)}|\nabla\, f(s)|ds\right]$$

When $m \to \infty$ we obtain

$$(4.19)\quad E^*(u(t))^{1/2} \leq C^*\left[E^*(u(0))^{1/2}e^{-\alpha\gamma\, t} + \int_0^t e^{\alpha\gamma(s-t)}|\nabla\, f(s)|ds\right]$$

Since $f \in L^1(0, T; H_0^1(\Omega)) \cap L^2(0, T; L^2(\Omega))$ by Lebesgue Convergence Theorem it follows

$$\lim_{t \to \infty} \int_0^t e^{\alpha\gamma(s-t)} |f(s)| ds = 0$$

$$\lim_{t \to \infty} \int_0^t e^{\beta\gamma(s-t)} |\nabla f(s)| ds = 0$$

therefore

$$\lim_{t \to \infty} E(u(t)) = \lim_{t \to \infty} E^*(u(t)) = 0$$

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