

Some fixed point theorems for Kannan mapping in the modular spaces

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Abstract

In this article, a new version of Kannan mapping theorem in modular space is presented. The main result of this paper is the existence of fixed point of Kannan mapping in complete modular spaces that have Fatou property.

Key words: *Fixed point, Kannan map, Modular space.*

AMS subject classification: 47H10, 46A19, 46B20, 47H09.

1 Introduction

In this article, existence of a fixed point of Kannan mapping in the modular space is proved. The theory of this space was initiated by Nakano [9] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [8] in 1959. In order to do this and for the sake of convenience, some definitions and notations are recalled from [2],[3], [4], [5], [7], [8], [9] and [10].

Definition 1.1 Let X be an arbitrary vector space over $K(= \mathbb{R}$ or $\mathbb{C})$.

A functional $\rho: X \rightarrow [0, +\infty)$ is called modular if:

1. $\rho(x) = 0$ if and only if $x = 0$.
2. $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$; $\forall x, y \in X$.
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$; $\forall x, y \in X$.

Definition 1.2 If 3 in Definition 1.1 is replaced by:

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y) \quad \text{for } \alpha, \beta \geq 0, \alpha^s + \beta^s = 1 \text{ with an } s \in (0, 1]$$

then the modular ρ is called an s -convex modular; and if $s = 1, \rho$ is called a convex modular.

Definition 1.3 A modular ρ defines a corresponding modular space, i.e. the space X_ρ is given by

$$X_\rho = \{x \in X \mid \rho(\gamma x) \rightarrow 0 \text{ as } \gamma \rightarrow 0\}.$$

Definition 1.4 Let X_ρ be a modular space.

1. A sequence $\{x_n\}$ in X_ρ is said to be:

(a) ρ -convergent to x if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow +\infty$.

(b) ρ -Cauchy if $\rho(x_n - x_m) \rightarrow 0$ as $n, m \rightarrow +\infty$.

2. X_ρ is ρ -complete if any ρ -Cauchy sequence is ρ -convergent.

3. A subset $B \subset X_\rho$ is said to be ρ -closed if for any sequence $\{x_n\} \subset B$

with $x_n \rightarrow x$, then $x \in B$. \overline{B}^ρ denotes the closure of B in the sense of ρ .

4. A subset $B \subset X_\rho$ is called ρ -bounded if:

$$\delta_\rho(B) = \sup_{x, y \in B} \rho(x - y) < +\infty,$$

where $\delta_\rho(B)$ is called the ρ -diameter of B .

5. We say that ρ has Fatou property if:

$$\rho(x - y) \leq \liminf \rho(x_n - y_n)$$

whenever

$$x_n \xrightarrow{\rho} x, \quad y_n \xrightarrow{\rho} y$$

6. ρ is said to satisfy the $\Delta 2$ -condition if $\rho(2x_n) \rightarrow 0$ as $n \rightarrow +\infty$,

whenever $\rho(x_n) \rightarrow 0$ as $n \rightarrow +\infty$.

Now, we state the definition of Kannan mapping in the modular space (see [6]).

Definition 1.5 $T: X_\rho \rightarrow X_\rho$ is called a Kannan mapping, if there exist

$$c \geq 1 \text{ and } \alpha \in \left[0, \frac{1}{2}\right), \text{ such that}$$

$$\rho(c(Tx - Ty)) \leq \alpha(\rho(Tx - x) + \rho(Ty - y))$$

for all $x, y \in X_\rho$.

Theorem 1.6 Let X_ρ be complete modular space which ρ satisfies the Fatou property. Suppose $T: X_\rho \rightarrow X_\rho$ be a Kannan mapping and assume that there exists $x \in X_\rho$ which $\rho(Tx - x) < \infty$. Then T has a unique fixed point.

Proof. By induction on $n \in \mathbb{N}$, we prove the following inequality,

$$\rho(c(T^{n+1}x - T^n x)) \leq \left(\frac{\alpha}{1-\alpha}\right)^n \rho(Tx - x).$$

Let $x \in X_\rho$, for $n = 1$ inequality holds.

Assume for $n = k \in \mathbb{N}$ we have

$$\rho(c(T^{k+1}x - T^k x)) \leq \left(\frac{\alpha}{1-\alpha}\right)^k \rho(Tx - x).$$

Now, for $n = k + 1$,

$$\begin{aligned} \rho(c(T^{k+2}x - T^{k+1}x)) & \\ & \leq \alpha(\rho(T^{k+2}x - T^{k+1}x) \\ & \quad + \rho(T^{k+1}x - T^k x)) \end{aligned}$$

Therefore

$$\begin{aligned} \rho(c(T^{k+2}x - T^{k+1}x)) - \alpha\rho(T^{k+2}x - T^{k+1}x) & \\ \leq \alpha\rho(T^{k+1}x - T^k x) & \\ \leq \frac{\alpha^{k+1}}{(1-\alpha)^k} \rho(Tx - x), & \end{aligned}$$

and since

$$\begin{aligned} (1-\alpha)\rho(c(T^{k+2}x - T^{k+1}x)) & \\ \leq \rho(c(T^{k+2}x - T^{k+1}x)) & \\ - \alpha\rho(T^{k+2}x - T^{k+1}x) & \end{aligned}$$

Therefore

$$\begin{aligned} (1-\alpha)\rho(c(T^{k+2}x - T^{k+1}x)) & \\ \leq \frac{\alpha^{k+1}}{(1-\alpha)^k} \rho(Tx - x), & \end{aligned}$$

and this shows that

$$\rho(c(T^{k+2}x - T^{k+1}x)) \leq \frac{\alpha^{k+1}}{(1-\alpha)^{k+1}} \rho(Tx - x).$$

Now, let $m > n$, then we have

$$\begin{aligned} \rho(c(T^n x - T^m x)) & \\ & \leq \alpha(\rho(T^n x - T^{n-1} x) \\ & \quad + \rho(T^m x - T^{m-1} x)) \\ & \leq \alpha \left(\left(\frac{\alpha}{1-\alpha}\right)^{n-1} \rho(Tx - x) + \left(\frac{\alpha}{1-\alpha}\right)^{m-1} \rho(Tx - x) \right), \end{aligned}$$

Letting n tend to infinity, then

$$\lim_{n \rightarrow \infty} \rho(c(T^n x - T^m x)) = 0.$$

Consequently, $\{T^n x\}$ is a Cauchy sequence. Then by completeness of X_ρ , there is a $z \in X_\rho$

such that $T^n x \rightarrow z$. Now, we show that $Tz = z$. By Fatou property we have

$$\rho(Tz - z) \leq \liminf_n \rho(Tz - T^n z)$$

Since

$$\rho(Tz - T^n z) \leq \rho(c(Tz - T^n z))$$

we obtain

$$\liminf_n \rho(Tz - T^n z) \leq \liminf_n \rho(c(Tz - T^n z))$$

$$\liminf_n \alpha(\rho(Tz - z) + \rho(T^n x - T^{n-1} x)) \leq$$

then

$$(1-\alpha)\rho(Tz - z) \leq \liminf_n \alpha(\rho(T^n x - T^{n-1} x))$$

If n tends to infinity we have $Tz = z$. Thus z is a fixed point of T and z is unique.

Otherwise, if $u \in X_\rho$ that $u = u$, $u \neq z$ then

$$\begin{aligned} \rho(z - u) & = \rho(Tz - Tu) \\ & \leq \alpha(\rho(Tz - z) + \\ & \quad \rho(Tu - u)) \\ & = 0 \end{aligned}$$

therefore

$$\rho(z - u) = 0$$

and this means that

$$z = u.$$

□

Definition 1.7 Let T be a map of X into itself, let $z \in X$. T is ρ -sequentially continuous at z , if and only if when $x_n \rightarrow z$, then $T(x_n) \rightarrow T(z)$.

Theorem 1.8 Let X_ρ be a modular space which satisfies $\Delta 2$ -condition. Let T be a map of X into itself and there exists a $c \geq 1$ such that:

$$(1) \quad \rho(c(Tx - Ty)) \leq \alpha(\rho(x - Tx) + \rho(y - Ty)); \text{ for } 0 < \alpha < \frac{1}{2}, x, y \in X.$$

(2) T is sequentially continuous at a point $u \in X$.

(3) There exists a point $x \in X$ such that the sequence of iterates $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ converging to u .

Then u is the unique fixed point of T .

Proof. T is sequentially continuous at u , therefore we have $T^{n_i+1} x$ converges to Tu .

By $\Delta 2$ -condition we have

$$\rho(c(T^{n_i} x - u)) \rightarrow 0; \rho(c(T^{n_i+1} x - Tu)) \rightarrow 0.$$

Suppose $u \neq Tu$.

Let

$$0 < \epsilon < \frac{1}{3} (\rho(\frac{1}{4} (c(u - Tu))),$$

then there exists a positive integer N , such that for $n_i > N$,

$$\rho(c(T^{n_i} x - u)) < \epsilon \text{ and } \rho(c(T^{n_i+1} x - Tu)) < \epsilon.$$

Now, we claim that

$$\rho(c(T^{n_i} x - T^{n_i+1} x)) > \epsilon; n_i > N$$

Because if

$$\rho(c(T^{n_i} x - T^{n_i+1} x)) \leq \epsilon$$

then

$$\begin{aligned} \rho\left(\frac{1}{4}(c(u - Tu))\right) &\leq \rho(c(u - T^{n_i} x)) + \\ \rho(c(T^{n_i} x - T^{n_i+1} x)) &+ \rho(c(T^{n_i+1} x - Tu)) \\ &< 3\epsilon \end{aligned}$$

and this is a contradiction.

On the other hand,

$$\begin{aligned} \rho(c(T^{n_i+1} x - T^{n_i+2} x)) \\ \leq \alpha(\rho(T^{n_i} x - T^{n_i+1} x) \\ + \rho(T^{n_i+1} x - T^{n_i+2} x)), \end{aligned}$$

Therefore

$$\begin{aligned} \rho(c(T^{n_i+1} x - T^{n_i+2} x)) \\ \leq \frac{\alpha}{1-\alpha} \rho(T^{n_i} x - T^{n_i+1} x) \end{aligned}$$

Let $n_k > n_j > N$, thus

$$\begin{aligned} \rho(c(T^{n_k} x - T^{n_k+1} x)) \\ \leq \left(\frac{\alpha}{1-\alpha}\right)^{n_k-n_j} \rho(T^{n_j} x \\ - T^{n_j+1} x) \end{aligned}$$

Let n_k tends to infinity, then we have

$$\rho(c(T^{n_k} x - T^{n_k+1} x)) \rightarrow 0$$

this is a contradiction, therefore $T(u) = u$.

If $z \in X$ be a point such that $Tz = z$ then

$$\begin{aligned} \rho(u - z) &= \rho(Tu - Tz) \\ &\leq \alpha(\rho(u - Tu) + \rho(z - Tz)) \\ &= 0 \end{aligned}$$

Hence $z = u$, and the theorem is proved. \square

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